A structured investigation of retirement income products

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Declaration

I hereby declare that this submission is my own work and to the best of my knowledge it contains no materials previously published or written by another person, nor material which to a substantial extent has been accepted for the award of any other degree or diploma at UNSW or any other educational institution, except where due acknowledgement is made in the thesis. Any contribution made to the research by others, with whom I have worked at UNSW or elsewhere, is explicitly acknowledged in the thesis.

I also declare that the intellectual content of this thesis is the product of my own work, except to the extent that assistance from others in the project's design and conception or in style, presentation and linguistic expression is acknowledged.

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ABSTRACT

Over the past years, there have been a wide range of retirement income products proposed in the academic literature and in industry. These include the traditional life annuity, as well as more innovative products which incorporate longevity risk sharing. However, the systematic comparison of these products is difficult due to the presence of differing guarantee and payout structures.

In this thesis, the differences in longevity and financial guarantee structures in various retirement income products are first compared through a modelling framework. This framework gives the payout of the product, taking into account the cost of any relevant guarantees. This is achieved using a fund equation, which mimics the reserve of the provider. Loadings are added to the equation to take into account the differences in the cost of providing longevity and financial guarantees. The payouts are then simulated by using stochastic mortality rates and financial returns. Second, an evaluation framework is developed to gauge the desirability of the products from the perspective of the policyholder and provider. A lifetime utility framework is used to provide a ranking of the desirability of each product. We also evaluate the product using a risk measure developed by the Australian Government Actuary (2018).

We find that the guarantee structure of the product makes a significant difference to both the capital required by the provider as well as the product desirability for the policyholder. In particular, across a wide range of economic and mortality scenarios, group self annuitisation is found to be the most preferred product due to its equity participation. The life annuity, on the other hand, is less preferred in most cases due to the high cost of its longevity and financial guarantee. The findings from this project have significant implications for both policymakers and retirement income product designers and providers.

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CHAPTER 1

INTRODUCTION

1.1 Motivation

Throughout the world, the population is ageing. This is due to increasing life expectancy and a decline in fertility rates. Globally, the number of people aged over 60 is expected to increase from 12% in 2015 to 22% by 2050 (World Health Organisation, 2018). This places a greater burden on governments to provide retirement benefits for the elderly. The problem of providing a sustainable and adequate retirement income is compounded by the decline in occupational defined benefit (DB) pensions in many countries, which have historically provided lifetime incomes with high replacement rates. DB pensions have been increasingly replaced with defined contribution (DC) systems, which provide for accumulated wealth through one's working life. Retirees must then convert this wealth into a sustainable income stream, while needing to address numerous retirement risks, including financial, longevity and health risks.

Financial risk is the risk that an individual's accumulated capital falls in value due to investment fluctuations. This risk can be controlled through the choice of investment strategy for individuals. For providers, they can implement hedging strategies to reduce their exposure to financial markets.

Longevity risk is broadly defined as the risk that an individual lives longer than expected, and hence outliving their financial resources. This risk can be reduced through sharing it with other retirees by purchasing a retirement income product from a provider – this is referred to as risk pooling. The provider then needs to quantify this risk, which can be split into two sub-types: *systematic* and *idiosyncratic* risk. The systematic risk is the risk that the mortality of the whole pool deviates from the provider's expectations. This risk arises because forecasts of future mortality are uncertain, and the provider does not know how mortality improvements will evolve over time. This risk is not diversifiable – it cannot be reduced by selling more policies. Idiosyncratic risk is the risk that the mortality of any individual deviates from expectations. This risk for each policy can be reduced by selling more policies, and is diversifiable.

Long term care risk is also very significant in retirement. It is the risk that the individual will not be able to meet aged care costs, which are uncertain in timing and magnitude. The modelling of health contingencies is complex and highly dependent on the institutional environment, including government provision of health insurance. In this thesis the focus will be on financial and longevity risk.

In the economic literature, the solution to managing the financial and longevity risk is a life annuity, which is found to be an optimal product to purchase in retirement (Yaari, 1965, Davidoff et al., 2005). This result holds under several key assumptions: that the consumer maximises their lifetime utility; that the return on annuities is higher than a comparable asset, such as a bond; and that the individual places no value on the wealth when dead. The life annuity can give higher returns than a bond because of *mortality credits*. Retirees who hold an annuity forfeit their wealth when they die, passing it on to the remaining surviving annuitants. Hence the surviving annuitants earn a higher return due to the longevity risk pooling.

However, the observed demand for annuities is low throughout most developed nations (Mitchell and Piggott, 2011). There are several reasons why this could be the case. One of the key reasons is that annuity providers need to charge a profit and capital loading, in order to meet the promises to the policyholders. This can make the annuity unattractive in relation to other investments. The bequest motive also makes annuities less attractive, since an annuity leaves no capital to the estate of the deceased.

Furthermore, there are numerous behavioural reasons which can affect how the embedded longevity guarantee in an annuity is perceived (Brown, 2008). For instance, loss aversion is a heuristic which states that people do not value the longevity guarantee in annuities because they are afraid of dying too early. The investment frame, which emphasises the annuity's lower returns, reduces the demand for an annuity compared to a consumption frame, which emphasises the constant lifetime income.

In the academic literature globally, there has been a discussion of several retirement income products which aim to address the high capital loadings required by annuities. Notably, these products aim to preserve the longevity sharing mechanism in life annuities, the mortality credits, which give rise to higher payments for survivors. Group self annuitisation (Piggott et al., 2005) is an arrangement where the longevity and financial risk is shared among the pool of participants, rather than being guaranteed by the provider. A longevity-indexed life annuity (Denuit et al., 2011) is where the systematic longevity risk is held by the annuitants, and the idiosyncratic risk is held by the provider. The tontine (Milevsky and Salisbury, 2015) is an arrangement where the provider promises a financial return, but leaves the longevity risk with the pool of participants. All these products aim to reduce the capital requirement, and hence the cost of the product, as compared to a life annuity, while still preserving longevity risk sharing arrangements.

In addition, there are also numerous features which can be added to a given guarantee structure. Capital guarantees return a portion of the capital to the annuitant if they die early in the contract, which overcomes the behavioural heuristic of loss aversion. Deferment periods can be introduced to make a product less expensive by deferring the longevity guarantee. Features such as these make the comparison more difficult by making the guarantee structure more complex.

In Australia, the problem of a lack of retirement income product development is similar. The proportion of the Australian population which is over 65 is growing at a rapid rate. According to the *2015 Intergenerational Report* (Commonwealth of Australia, 2015), it is expected to grow from 15% in 2015 to 23% in 2055. To respond to this challenge, the Australian Government has formalised a three-pillar retirement income system to respond to this challenge, which has been endorsed by the World Bank report *Averting the old age crisis* (The Treasury, 2001). This three-pillar system comprises of the first-pillar safety net of the Age Pension, the second-pillar DC system of superannuation, and the third pillar of voluntary saving.

The second-pillar DC system of superannuation was introduced in 1992, designed to 'supplement' the Age Pension by providing for retirement income through legislated mandatory saving of 9.5% of a worker's income (The Treasury, 2016). This money is invested in a superannuation fund and earns returns each year. As the system slowly matures, it is becoming a dominant part of Australia's retirement income system. Upon retirement, however, there is little guidance on how to best manage this wealth accumulation.

There is evidence to suggest that retirees in Australia do not manage their longevity and financial risks well. In 2017, 93% of Australian retirees who choose an income stream convert their superannuation balance into an account-based pension (ABP) (CEPAR, 2018), which is an example of a phased withdrawal product. An ABP is designed to give flexibility to consume, subject to mandated minimum drawdown requirements, which increase with age. However, the

ABP does not offer any protection against financial, longevity or health risk, being essentially a form of self-insurance. In response to the lack of formal risk management, 44% of retirees draw down at the minimum rates, which act as a hedge to probable future adverse outcomes (Balnozan, 2018, Asher et al., 2017). Life annuities, despite being the optimal products to manage longevity and financial risk, remain only a small proportion of total annuity sales (CEPAR, 2018). This can be partially explained by the fact that the Age Pension acts as a form of life annuity, and so reduces the demand for annuities from the private market (Iskhakov et al., 2015).

The lack of development of annuities and other retirement income products in Australia indicates that the 'retirement income system is underdeveloped', according to the Commonwealth of Australia (2015). The Australian Government has sought to address this by introducing the concept of a Comprehensive Income Product for Retirement (CIPR) (Commonwealth of Australia, 2015). This is a composite product which incorporates flexibility to access a lump sum, a higher income than an account based pension, and a broadly constant income for life. Income flexibility is important in addressing health and other contingency risks, while longevity risk management will provide a high, but broadly constant income for life. The product may not fully guarantee longevity risk protection, however, making it cheaper than a life annuity. The development of composite products such as the CIPR is an important step to catering for retirees' diverse needs.

Despite this innovation, there have been limited attempts to standardise the notation, modelling and comparison of retirement income products and features. The assumptions behind the comparison of each product, such as the interest rate, mortality rate and time interval of payments are markedly different and this obscures the commonalities of the underlying design of the product.

In this thesis, we aim to overcome this by first developing a mathematical modelling framework based on the concept of a *fund equation*, which represents the reserve required to be held to guarantee the payments promised to a policyholder. Pitacco et al. (2009) have developed the fund equation for a life annuity, and state the mortality credits, financial returns and payout structure as key elements of the modelling of an annuity. We aim to extend this concept to other retirement income products, incorporating changes to the financial and longevity guarantee structure. Part of the modelling framework will be dedicated to comparing the riskiness of products, and hence the capital charge required, for a wide variety of guarantees. This will assist insurers as they consider a variety of product designs in the rapidly developing universe of retirement income products.

Our companion evaluation framework will contribute to the comparison of retirement income products, which will provide practical insights to retirees. This will be achieved through providing a consistent comparison of the distribution of benefit payments, taking into account the riskiness in providing the guarantees. We also gauge the overall value of each product through utility measures. We expect such an evaluation framework will aid in the communication of these results to the government and industry. Indeed, the Retirement Incomes Working Group of the Actuaries Institute in Australia is also looking to analyse the features of retirement income products in the Australian context to provide policy input from an actuarial perspective (Asher and Swinhoe, 2019).

1.2 Research aims

To summarise, our research aims to provide a comprehensive and structured modelling and comparison of retirement income products. We aim to achieve the following research goals:

- To develop a mathematical framework to represent the guarantee structure in retirement income products, and;
- To comprehensively evaluate the value of such products from the perspective of both the insurer and policyholder.

1.3 Outline of thesis

The remainder of the thesis is set out as follows. Chapter 2 considers in detail the literature surrounding retirement income products, with an emphasis on the products which are examined in the modelling and evaluation frameworks. We also consider the academic literature on the comparison of financial and longevity guarantees. Chapter 3 describes the methodology on the development and implementation of the fund equation for a wide variety of retirement income products. We also consider each element of the modelling and evaluation framework in further detail. Chapter 4 reports the key results, focusing on the evaluation of the product from the insurer's and policyholder's perspectives. Sensitivity analysis is also performed on the key parameters of the model. Chapter 5 concludes the thesis. It describes the implications for policymakers, industry and academia. We conclude with suggestions for future work.

CHAPTER 2

LITERATURE REVIEW

This chapter will first review the existing literature around retirement income products. These products are grouped into three main categories. First, the products with longevity and financial guarantees which are fixed at inception of the contract will be described. Second, the products which incorporate the sharing of longevity and/or financial risk between provider and individual will be described. Finally, products which pool longevity risk are compared. These products do not have any provider guarantees of longevity risks, as these risks are shared among participants. The chapter will conclude with a brief discussion on the comparison of the products' guarantee and payout structure, with an emphasis on developing a common framework for their representation.

2.1 Retirement income products

2.1.1 Products with fixed guarantees

2.1.1.1 Life annuity

The life annuity is a product which pays a guaranteed flat periodic payment, as long as the retiree is alive. This product guarantees all financial and longevity risk. Yaari (1965) showed that a utility-maximising consumer under the standard lifecycle model, in the presence of complete markets and actuarially fair annuities, would fully annuitise. Davidoff et al. (2005) extends this result, finding that under complete markets, full annuitisation is still optimal even with non-actuarially fair annuities, so long as the rate of return on annuities is greater than the return on conventional assets of comparable risk. However, in practise, the loadings on annuities are high due to adverse selection, which make them less desirable.

2.1.1.2 Deferred life annuity

A variant on a standard life annuity is a deferred life annuity (DLA), which is characterised by a deferment period. If an individual survives beyond the deferment period, which is agreed between the provider and the individual upon inception, they receive a constant periodic benefit for life. The amount of this benefit is also determined when the product is purchased (Pitacco, 2016). Milevsky (2005) has introduced a variation of a DLA called an Advanced Delayed Life Annuity (ALDA). It incorporates a number of annual premium payments throughout one's working life in exchange for known periodic payments from an advanced age, say 80 or 90. The key guarantee structure is identical in these products: the financial guarantee is set at inception and lasts throughout the lifetime of the individual; the longevity guarantee applies from the conclusion of the deferment period until the death of the individual. This product has the potential to address longevity risk where it is most acute – in later years where the risk of running out of one's savings is at its highest.

2.1.1.3 Annuities with capital guarantees

Another variation is the presence of a capital guarantee in annuity products. This feature is variously referred to as value protection, money-back or cash-back (Pitacco, 2016). In one such setting in Boardman (2006), the money-back annuity is defined such that the original capital is returned upon death to the annuitant, minus any payments made so far in the contract. The concept of loss aversion and mental accounting can explain the presence of these additional guarantees. An individual may frame the decision to purchase an annuity as a gamble, with the payoff being random, depending on the annuitant's estimate of their lifetime (Brown, 2008). Brown (2008) suggests that under cumulative prospect theory, the losses of dying young outweigh the gains from living to an old age. The capital guarantee addresses this behavioural heuristic, as at least the initial capital is always returned to the annuitant. A variation on this idea has been developed in the Australian context (Comminsure, 2017, Challenger, 2019), where a known percentage of the initial capital is returned, if the annuitant dies in earlier years. Another variation is where the reserve of the individual's fund is paid out upon death. The presence of a capital guarantee adds a layer of complexity to the guarantee structure; there is not only a survival benefit, but also a possible benefit paid upon death. It therefore reduces the longevity risk faced by the provider.

Annuities can also be sold with other types of guarantees. A joint life annuity is a type of annuity where the payment is guaranteed as long as one out of two people covered under the policy are still alive. A reversionary life annuity is where the payment continues to a nominated beneficiary, most commonly, the spouse, after the annuitant dies. Finally annuities are commonly sold with inflation protection, called indexed annuities.

2.1.2 Products with partial guarantees

In lieu of providing a full financial and longevity guarantee, providers and academics may design innovative guarantee structures which share risk between the provider and the individual.

2.1.2.1 Variable annuity

The most notable of such products is the variable annuity. A variable annuity is a type of investment product where the policyholder invests a lump sum or periodic stream of payments, and in return the insurer can guarantee a wide variety of survival or death benefits. One example of such an survival benefit is the guaranteed minimum income benefit. This is where the provider offers the insured with a choice to purchase a whole life annuity: either at prevailing rates, or at guaranteed rates, which have been set at inception (Bauer et al., 2008). In this case the financial risk is shared by the individual and the provider during the deferment period. Under this guarantee, the longevity risk is borne by the individual during deferment, and by the provider after deferment. The simplest type of guaranteed minimum death benefit function ensures capital protection during the deferment period, where the maximum of the reserve and the initial premium is paid upon death (Bauer et al., 2008). Here, the financial risk is again shared between the provider and the individual, whereas the longevity risk is reduced for the provider due to the presence of the offsetting death benefit. Both of these guarantees provide the annuitant with options which can be exercised when financial markets deliver lower than expected returns. The cost of offering such guarantees has been quantified using complex option pricing models (Ignatieva et al., 2016, Alonso-García et al., 2018).

Related to the idea of financial risk-sharing between provider and policyholder is the notion of risk smoothing. Annuity products could be designed so that surpluses earned in good years support benefit payouts in bad years (van Bilsen and Linders, 2019). As the thesis will mainly revolve around the conceptual analysis of longevity risk pooling, rather financial risk-sharing, we decide not to analyse these further.

2.1.2.2 Longevity-indexed life annuity

A recent innovation in the development of longevity risk-sharing comes in the form of a longevity-indexed life annuity (Denuit et al., 2011). This product allows the payment at time t to fluctuate according to a longevity index, which is a *forecast* of mortality at time t based on a reference population, ${}_tp_x^{ref}$, made and agreed at inception between the annuitant and the provider. This fluctuation can be quantified through the following benefit equation for a payment at time t: $b_t = b_0 \times A_t$, where A_t is:

$$A_t = \frac{{}_tp_x^{ref}}{{}_tP_x}, \tag{2.1}$$

where ${}_tP_x$ denotes observed survival probabilities from age x to age $x + t$.

If the actual longevity exceeds that of the forecasted index, as in the case of unanticipated mortality improvements, then the payment will be reduced accordingly, and vice versa. In the case of no deviation of mortality from the forecasted index, naturally, this reduces to the case of a simple life annuity. Therefore, the guarantee structure can be stated as follows: the idiosyncratic mortality risk from year to year is the responsibility of the provider, as is the financial risk. However, the systematic risk is held by the pool of annuitants. This means that the provider can absolve themselves of the systematic component of their longevity risk. This is likely to result in lower capital requirements, and thus, a cheaper product.

The complete transfer of systematic longevity risk back to the annuitants can be undesirable, especially at older ages, where the actual survival of the annuitants is likely to differ significantly from the forecast. Therefore, floors and caps can be developed which restrict the deviation in payment between forecasted and actual mortality. In other words, the ratio in equation 2.1 is bounded by set constants $(A_{\min}, A_{\max})$. Here, the guarantee structure changes slightly, to allow the systematic risk to be taken by the pool only within the bounds, with the provider taking the remaining risk, which is outside the bounds.

2.1.2.3 Longevity-contingent deferred life annuity

Alternatively, the deferment period can fluctuate according to systematic improvements in mortality, as proposed in Denuit et al. (2015). A product under this arrangement is called a longevity-contingent deferred life annuity (LCDLA). This can be combined with the payment fluctuation as outlined above in Denuit et al. (2011). Initially, at time 0, age x, a best-estimate deferment period of m years is set. At the end of the deferment period, at time $t + m$, the life expectancy of a $x + m$ year old given by a government agency is compared to some arbitrary life expectancy agreed at inception, termed the 'threshold' life expectancy. If at time m, this life expectancy is less than the threshold, then there is no deferment. If this life expectancy exceeds the threshold, the additional deferment period is chosen so that it shares systematic longevity risk between the provider and individual.

It can be shown that if the threshold life expectancy is set at the forecast of the life expectancy of a $x + m$ year old, the provider is absolved of any unanticipated improvement in mortality from time 0 to time m. Under this arrangement, the systematic mortality risk is wholly borne by the individual from time 0 to time m. The idiosyncratic mortality risk and the remaining systematic portion after time m is still borne by the provider, along with the financial risk.

Denuit et al. (2015) use period life expectancy to illustrate their arguments, as this is the most common type of life expectancy given by government agencies. Alternatively, cohort life expectancy could be used instead. As cohort life expectancy takes into account improvements in mortality throughout the individual's life, the risk sharing arrangement changes too.

There are certain limits, as with the longevity-indexed life annuity, that the provider can set, such that the annuitants do not absorb too much of the systematic mortality risk. This can be easily done by incorporating a cap on the deferment length.

The advantages of the product are similar to that of a longevity-indexed life annuity; there are also additional innovations. First, for retirees, the use of a life expectancy measure is more likely to be understood than a longevity index. Second, for providers, the deferred nature of this product means the guarantees apply over a shorter period, further reducing the capital requirements and cost.

2.1.3 Products with pooled benefits

Alternatively, no guarantees at all can be explicitly provided by an insurer; instead, the financial and longevity risk can be shared among participants. In the simplest case, an insurer can sell a product to a group of individuals who are all of the same age and initial wealth. We refer to these individuals as the *pool*. The individuals who decide to participate cannot leave the pool after it has been set up and forfeit their share of the wealth upon death. This implies that the provider is able to redistribute their wealth to surviving members. Hence, the benefits paid to survivors depend on the mortality experience of this pool as a whole, and possibly the investment strategy of the pool. It is important to note that such products cannot provide any protection from systematic mortality improvements, as the provider does not hold any capital to guarantee this risk.

2.1.3.1 Group self annuitisation

Perhaps the most well-known example of product with pooled benefits is the group self annuitisation (GSA) arrangement (Piggott et al., 2005). The provider pools wealth from participants and invests it according to their investment strategy. In return, participants receive a regular income determined by the provider. Participants forfeit their wealth in the fund upon death, similar to a life annuity where there are no payments after death. The arrangement is designed, so that in expectation, the payments would function like a life annuity, but without any guarantee. Therefore the pool shares the idiosyncratic longevity risk among itself, while also bearing the systematic longevity risk and financial risk. This implies that the provider need not hold any capital for mortality or investment risk.

The payout structure is as follows: assume that l_x annuitants, each with the same wealth and age x, decide to enter into a GSA at time $t = 0$. The payments would be adjusted to take into account the actual number of lives alive and the investment performance at each time t. Piggott et al. (2005) show that this adjustment for mortality and investment would be:

$$b_t = b_{t-1} \times MEA_t \times IRA_t, \tag{2.2}$$

where the mortality experience adjustment, $MEA_t = \frac{p_{x+t-1}}{P_{x+t-1}}$ and the interest rate adjustment, $IRA_t = \frac{r_t}{R_t}$. Capital letters denote 'actual' or 'realised' quantities. In the MEA term, p_{x+t-1} (P_{x+t-1}) denotes the assumed (actual) survival probability from time $t-1$ to time t. In the IRA term, r_t (R_t) denotes the assumed (actual) financial return. This formulation implies that the mortality and interest adjustments only apply as experience emerges in the contract. No attempt is made to determine where there is a secular shift in mortality or interest rates in future periods.

Piggott et al. (2005) also discuss more complex cases, including the entrance of new cohorts and expectations adjustment. They show that new cohorts can be incorporated in a fair way – the pool does not need to be closed after the initial pool is established. They calculate a modified mortality and interest rate adjustment factor which takes into account the deviations in mortality, but only for those cohorts which have entered the pool in the previous periods before the deviation occurred. That is to say, if the deviation in mortality occurred between time $t-1$ and time t, only those people who have entered the pool between time 0 and time $t-1$ would be affected. If the risk of this deviation is shared equally across all such cohorts, the authors show this will lead to reduced variability in payments – a clear advantage, and may lead to more entrants in the future.

The concept of expectations adjustment is motivated by the case of a secular shift in mortality and/or interest rates. This is an example of systematic longevity risk applying to the whole portfolio. Piggott et al. (2005) model this by using a new annuity factor to calculate future benefits. This factor applies starting from a given time t, to a single cohort of individuals. The authors show that a one-time adjustment to the benefit payments must be made at time t to this cohort to incorporate this information. Assuming no further deviation of mortality from this new basis, no adjustments need to be made beyond this time and therefore the benefit payment is flat thereafter. This type of permanent adjustment to the benefit payment stands in contrast to the gradual incorporation of information in each period as in equation 2.2, since it is prospective rather than retrospective. Due to this foresight, it minimises the probability of fund exhaustion at very old ages.

However, the method of incorporating systematic mortality risk is crude, and the effect of pool size is not considered. Qiao and Sherris (2013) remedy this shortcoming by incorporating systematic mortality risk in a dynamic fashion to multiple cohorts. They show that without taking into account systematic mortality improvements, the benefit payments will be declining over time. Qiao and Sherris (2013) then attempt to solve this problem by designing a new pooling scheme, which incorporates the systematic trend of mortality improvements in the annuity factors dynamically using inputs from a stochastic version of the Gompertz-Makeham mortality model. The pooling scheme is also extended to multiple cohorts. They allow different cohorts to share the systematic risk, thereby reducing the volatility of payments. The effect of pool size on the variability of payments is also considered. The investment risk, however, is only incorporated in a superficial way. Throughout most of the analysis, a flat yield curve is used, following Piggott et al. (2005). Only in the latter part of the analysis is the CIR model used to simulate a stochastic interest rate.

2.1.3.2 Pooled annuity fund

Stamos (2008) modify the work of Piggott et al. (2005), proposing a new type of pooled contract called a pooled annuity fund (PAF). This fund pools contributions and invests them in a risky asset, with the longevity risk being shared among the pool, similar to the guarantee structure of a GSA. Individuals also forfeit their wealth upon death. However, unlike a GSA, members have freedom in consumption, instead of it being dictated at inception by the provider. For the purposes of modelling the product, it is assumed that individuals enter into a PAF with the same wealth and same mortality, and follow the same investment strategy. Members receive mortality credits, which are the shared gains from other members' deaths equally distributed among all survivors. These are credited instantaneously upon the j^{th} individual's death at a rate of:

$$\frac{1}{L_{t-} - 1} dN_t \tag{2.3}$$

where N_t is a Poisson process describing the evolution of deaths and L_{t-} is the number of lives just before the j^{th} individual has died. The mortality credit is stochastic in timing and magnitude since L_{t-} is constantly evolving and the time of any single death is uncertain. Stamos (2008) then optimises for the consumption path which maximises lifetime utility. This consumption path is increasing through time due to the high mortality credits gained at older ages. This paper contributes to the literature in two important ways. First, a new payout structure is developed where the individual is free to consume. Second, the impact of investment risk on the optimal consumption choice is analysed.

The design of the PAF has been extended by Mercer to accommodate individuals with differing wealth and mortality, who enter the pool at different times. This product was known as Mercer LifetimePlus (Mercer, 2017), and is not offered by Mercer at present. Similar to the PAF, it was designed as an investment-linked account, which provided mortality credits, referred to as the 'living bonus'. This 'living bonus' was funded by participants who withdrew from the fund or died, and passed on to the remaining survivors in the fund. Participants would consume the investment returns from the fund, and would also receive a return of their capital if they stayed in the fund for more than twelve years. The consumption freedom and mortality credits are the key similarities to the PAF. The return of the retiree's capital and the ability to leave part of the wealth as a bequest upon death are distinguishing features.

2.1.3.3 Mortality-linked fund

Donnelly et al. (2013) introduce a new product called the mortality-linked fund (MLF), which bears some similarities to a PAF. Once again, the fund pools contributions to invest them in a risky asset and individuals forfeit their wealth upon death. However, instead of the mortality credit being paid stochastically as in Equation 2.3, it is paid according to

$$\mu_{x+t}(1-\delta_t)dt$$

where μ_{x+t} is the deterministic force of mortality at age $x+t$ and δ_t represents the costs to the provider. These costs are incurred since the longevity risk is taken up by the provider, and not the individual. There is no longer any volatility, in the magnitude or the timing of the mortality credit. Accordingly, the only risk that the individual bears is financial risk which could deplete the value of their fund. Similar to the assumption in the PAF, individuals enter into a MLF with the same wealth and same mortality, and follow the same investment strategy.

A similar product design is a unit-linked annuity (Wadsworth et al., 2001, Asher and Swinhoe, 2019). This product, like the MLF, offers unitholders investment in a fund which could comprise of risky or safe assets, while promising them a deterministic mortality credit. Here, the assumption of individuals entering such an arrangement with the same wealth and mortality is loosened. Varying designs are detailed in Wadsworth et al. (2001), with some products offering smoothed returns (known as with-profit annuities), investment choice in the type of fund that is chosen, and allowing changes in investment strategy over the lifecycle. In both the MLF and unit-linked annuity, consumption freedom can be offered, but in practice the consumption is likely to be set at moderate levels, to prevent withdrawal from the fund before death and associated adverse selection issues.

2.1.3.4 Annuity overlay fund

Donnelly et al. (2014) introduce the annuity overlay fund, a further modification of the PAF. This fund pools wealth in an investment fund, with longevity risk sharing between participants. This is contrast to a MLF, where the longevity risk is guaranteed by the provider. The payout structure of the annuity overlay fund is designed so that it is actuarially fair at any instant. That is, for any individual, at any instant, the expected gain of the mortality credit given to the surviving individual from other individuals who have died, is exactly equal to the expected loss of the individual's wealth upon death.

As a result of this fairness, individuals can join with differing wealth. Individuals are also free to leave the fund at any time, resulting in further flexibility. In practice, however, limits would need to be set on consumption each year to prevent adverse selection, as individuals in deteriorating health have an incentive to exit. The limit on consumption would also need to apply for this reason to a MLF or PAF.

2.1.3.5 Tontine

A different type of product is the tontine, a historical product recently revived in Milevsky and Salisbury (2015). Historically, this product was structured to pay out a predetermined dollar amount, $\mathcal{B}_t = X$, annually to all pool members, in other words, to the pool as a whole. This structure will be referred to as a *flat tontine*. This dollar amount was financed from the investors pooling their money together to purchase a long stream of bond coupons, with the principal payment funded by the custodian. Thus this meant that there was (little to) no financial risk, but the longevity risk was shared equally among the participants. In other words, a flat tontine shares a constant amount X equally among a pool of survivors L_t; with each survivor receiving their individual benefit $b_t = X/L_t$ every year. Notice that L_t is diminishing as the pool dies. This meant that at very old ages, the benefit X/L_t will increase rapidly, with last few survivors receiving a windfall. Intuitively, the flat tontine is far from optimal. Milevsky and Salisbury (2015) propose an adaption to the individual benefit b_t, to make it smoother over the lifetime.

Milevsky and Salisbury (2015) derive the continuous-time payout function for the *optimal tontine*, which is calculated so as to maximise the constant relative risk aversion (CRRA) utility function. The formulation for the payout is complex, but a simpler formulation can be developed which gives a broadly similar payout. This is called the *natural tontine*. Milevsky and Salisbury (2015) show that this payout approximates the optimal tontine at younger ages, with the payouts diverging significantly at very old ages. It is given by:

$$\mathcal{B}_t = {}_tp_x c_0$$

where c_0 is given by the flat payout function of the life annuity, $1/\left[\int_0^\infty e^{-rt}{}_tp_x dt\right] = 1/\overline{a}_x$ and ${}_tp_x$ is the survival probability from age x to age $x+t$. An equal portion of this payout is distributed to each surviving member of the pool.

2.1.4 Products which address LTC risk

So far, the discussion has concentrated around longevity and financial risk. Long term care (LTC) risk can be addressed through the development of products with benefits which pay out upon disablement of the individual. The seminal paper in this field is in Murtaugh et al. (2001), who design a life annuity with LTC benefits, called a *life care annuity*. They claim that the presence of natural hedging will reduce adverse selection in the life annuities, while reducing the requirement for medical underwriting in LTC insurance. Through simulation, they find that the disability costs make up a small proportion of the total payments, and the theoretical market for such a product is higher than that of a stand-alone annuity or stand-alone LTC insurance. The modelling behind such products has been extended in Levantesi and Menzietti (2012), who incorporate stochastic transition rates. This is further developed in Brown and Warshawsky (2013), who separate the population into eight different risk classes. Upon pooling healthy and moderately disabled lives, they find that the healthy people in the pool cross-subsidise the moderately disabled. However, in all cases, the presence of adverse selection and strict underwriting in the life annuity and LTC insurance, respectively, means that the policyholders would purchase a life care annuity rather than the two policies separately.

2.1.5 Hybrid products

The development of hybrid, or composite products is an important step in addressing retirees' holistic needs. A hybrid product, as the name suggests, combines the guarantee structures of the various products introduced earlier in Subsections 2.1.1 to 2.1.3. As stated in the Introduction, the Australian Government has recently sought to introduce the CIPR, a type of hybrid product. In The Treasury (2016, Chart 5), an example of such a CIPR is 'the wrap', a combination of an ABP and a DLA, with the ABP being drawn down faster in earlier years. In later years, what is remaining of the ABP is drawn down at minimum rates and the DLA provides the majority of the income. The time of the inception of the DLA is naturally fixed at the purchase of 'the wrap'. This product has several advantages compared to either stand-alone product. In earlier years, flexibility is very important as discretionary consumption, for instance, spending on holidays, peaks during this time (Daley et al., 2018). Health risks are also partially addressed by having an ability to take a lump sum in earlier years. In later years, a DLA is appropriate to provide efficient longevity risk protection.

Hybrid products can also be specifically tailored to health risk at older ages. Weinert and Gründl (2016) consider the case of a liquidity function to describe the liquidity needs at old ages which rapidly increases at older ages. The authors then describe the flat tontine (see Subsection 2.1.3), which has a increasing payout to individuals over time. They find that purchase of a combination of a flat tontine and an annuity at retirement can help retirees address their financial, longevity and health risks holistically. However, they also find that the optimal allocation to a tontine is only 12%. This result assumes actuarial fairness for the tontine and annuity. Thus, if loadings are applied to the annuity, the optimal allocation to the tontine could be even higher.

Other innovations based on the tontine and annuity are possible. One such example is a 'tonuity', introduced in Chen et al. (2018). This is a product that in earlier years has a natural tontine payout structure modelled on Milevsky and Salisbury (2015), while in later years gives a DLA payout structure. For the annuitant, the longevity risk is moderated through the tontine in earlier ages, while eliminated in later ages. Similar to the CIPR example, the time of inception of the DLA is fixed at purchase of the 'tonuity'. In Bernhardt and Donnelly (2019), the idea of a tontine with a bequest motive is introduced, where the funds are split between a tontine account and a bequest account. The proportion of money which is allocated between the two accounts is decided at inception, and the accounts are re-balanced to maintain this proportion over time. This can also be thought of as an extension to partial annuitisation where the retiree decides what proportion of their wealth is annuitised. However, here, the automatic rebalancing mechanism explicitly aligns the amount of bequest with the strength of the bequest motive.

2.2 Modelling framework

In this section, we describe the scholarly literature on the comparison of the guarantee and payout structures of retirement income products. We also consider the issue of loadings when there is a longevity or financial guarantee embedded in the product.

A key contribution in this space is in Pitacco et al. (2009), who approached this problem from the perspective of calculating the reserve of a life annuity in discrete time. Through developing a fund equation, they separate the gains from longevity risk pooling and financial returns for this product. However, they do not consider how this approach could be extended to other products, nor a method to quantify the cost of providing the financial and longevity guarantee.

In contrast, Hanewald et al. (2013) compare ten portfolios of retirement income products comprised of life annuities, GSAs, phased withdrawals and hybrid products. The first two products are a life annuity and a inflation-indexed life annuity. A stand-alone GSA and phased withdrawal (ABP) are considered next. The analysis of hybrid products follows – various combinations of a life annuity, ABP, GSA and DLA are considered, with different drawdown rates for the ABP. Hanewald et al. (2013) then simulate values of the portfolios and drawdown patterns of the ABP under the assumption of no financial risk. The simulated prices of the life annuity and DLA are each loaded by 10% and 25% to account for the financial and longevity risk faced by the providers. The authors then rank the portfolios under a utility framework. The 25% loading is taken from Ganegoda and Bateman (2008), an empirical Australian study. The desirability of the portfolios change when the life annuity is loaded, but there is little systematic justification for the changes. For example, the ordinary LA is second most preferred in the case without loadings, while it is seventh preferred in the case of a 25% loading. There is little change, however, in the ranking of the inflation-indexed life annuity. It is clear that a comparison between the portfolios would be aided by a more comprehensive examination of both the nature and cost of the financial and longevity guarantees in each portfolio.

Milevsky and Salisbury (2015) extend the investigation of longevity guarantees in annuities, particularly finding the cost of the longevity guarantee which makes an individual indifferent between a tontine and a loaded annuity. Similar to Hanewald et al. (2013) they focus on the loading as a percentage of the purchase price of an annuity, finding that this loading, even for very risk-averse individuals, is less than 10%. This means that the tontine could actually be preferred to an annuity when the present value of the annuity is 10% higher than a fair annuity. Similar analysis is performed in Donnelly et al. (2013) to find the cost required to guarantee the longevity risk in a mortality-linked fund versus an annuity.

Chen et al. (2018) extend this analysis by considering more realistic longevity risk charges under the European Solvency II regulations. They calculate the risk margin required to be paid by the policyholder for transferring longevity risk under various cost of capital assumptions. Since an insurance company is risky, the cost of capital can be quite high. The authors have assumed a 6% cost of capital for the base case. This means the loading on the premium the policyholder pays for the annuity is more realistic, as it is calculated according to a regulatory capital model rather than being an arbitrary parameter such as an indifference loading.

The analyses discussed above, however, do not consider a number of important aspects of the modelling in a unified manner: the separation of the gains from longevity risk pooling and financial returns; a systematic way to model and incorporate loadings in products with complex guarantee and payout structures; and differing payout structures where the products have the same longevity guarantee structure. In other words, various products, such as group self annuitisation, pooled annuity funds and tontines have remarkably similar design of the longevity risk pooling, however the method of paying out surviving members is quite different. The focus on the benefits given to retirees obscures the underlying commonalities in their longevity risk sharing.

One of the major contributions of this thesis, then, is to develop a unifying framework to capture the wide variety of financial and longevity guarantees in retirement income products. We extend the framework of Pitacco et al. (2009) to consider the reserve required by other products with varying guarantee structures. We develop a similar framework to Chen et al. (2018) to quantify the riskiness of products and calculate their required prices in a consistent manner. Finally, we build upon the evaluation of Hanewald et al. (2013) by incorporating a wider variety of risk and utility measures.

CHAPTER 3

METHODOLOGY

This chapter focuses on the techniques used to model and evaluate retirement income products. First, we introduce some notation which will be used throughout the thesis. Then, we introduce the various financial and mortality models which are used as inputs to the modelling framework. We next develop further the fund equation proposed in Pitacco et al. (2009), to incorporate a wide variety of guarantee structures and product features. This equation states the required reserve to be held for a policyholder who purchases a retirement income product. The elements in this equation, namely the mortality credits, financial return and payout structure, make explicit the longevity and financial guarantees present in retirement income products. Next, we consider the riskiness of the products, by calculating the required capital and associated price charged to the policyholder. Finally, we explain our approach to evaluating the retirement income products. This process can be summarised by Figure 3.1.

3.1 Notation

We assume that the product under consideration is taken out at age x, at time $t = 0$. We will assume all individuals who purchase the product have identical wealth and mortality, so that at time t, the individuals will all be aged $x + t$. We will now define some notation which will be used in the following sections.

We first define the terms in the fund Equation (3.9) in Section 3.3:

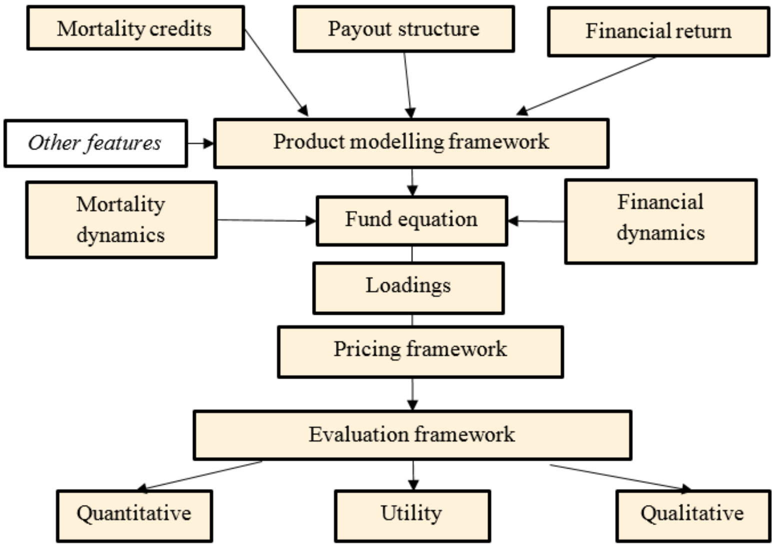

Figure 3.1: Modelling framework

- S: the initial investment used to purchase the product without loadings,
- S^*: the initial investment used to purchase the product, after deducting loadings,
- F_{t-}: the fund value for an individual at time t, contingent on survival to time t, *before* the benefit for that period has been paid out,
- F_t: the fund value for an individual at time t, contingent on survival to time t, *after* the benefit for that period has been paid out,
- Θ_t: the mortality credit earned between time $t-1$ and t,
- R_t: the stochastic financial return earned between time $t-1$ and t,
- r_t: the deterministic financial return earned between time $t-1$ and t,
- $b_t = F_{t-} \times c(t;\tau)$: a formula giving the total benefit payable to an individual alive at time t, written as a function of the fund value at time t, conditional on information available up to time t. $c(t;\tau)$ can be thought of as the rule, determined at time $\tau < t$, which states how much of the fund value is paid out at time t.
- I_t: a formula giving the total benefit payable at time t to an individual who has died between time $t-1$ and t.

We next define some variants of standard actuarial notation:

- l_{x+t}: the best estimate of the number of lives alive in the annuitant population at time t, made at time 0,
- L_{x+t}: the actual number of lives alive at time t,
- l_{x+t}^{ref}: an estimate of the number of lives alive in the reference population at time t, updated at each time t,

- ${}_hp_{x+t} \equiv l_{x+t+h}/l_{x+t} \equiv 1 - {}_hq_{x+t}$: standard actuarial notation for probability of survival of a life aged $x + t$, between time t and $t + h$, using the best estimate of the annuitant lives,
- $v_t \equiv 1/(1 + r_t)$: standard actuarial notation for the discount factor,
- $\ddot{a}_{x+t}$: standard actuarial notation for an annuity due starting at age $x + t$,
- $\mu_{x,t}$: the force of mortality for a person aged x at time t,
- ω: the maximum lifetime of an individual.

3.2 Model specification

This section outlines the models which are used to account for longevity and financial risk in a retirement income product.

3.2.1 Mortality modelling

There is a wide range of literature on mortality modelling. When modelling retirement income products, the performance of the model at ages older than 60 is paramount, since these ages are relevant for the payouts of the products. Among the earliest was the Gompertz-Makeham law of mortality (Makeham, 1860), who modelled mortality rates using a deterministic formula at a single point in time. The force of mortality at age x, μ_x is given by:

$$\mu_x = \frac{1}{b} e^{\frac{x-m}{b}},$$

where $b > 0$ is a dispersion parameter and m is the modal age at death. This model is very tractable and has been often used to determine closed-form solutions (see Milevsky and Salisbury, 2015, Chen et al., 2018). Additionally, it performs well at old ages.

Mortality variability at future times can be incorporated into a deterministic model through Monte Carlo simulation. One example of such an approach can be found in Piggott et al. (2005):

$$q_x^* = q_x \left[\frac{x}{100} [U(0,1) \times 0.3 - 0.15] + 1 \right] \tag{3.1}$$

where $U(0, 1)$ is a uniform (0,1) random variable and q_x is the input death probability from a certain population. This formulation has the consequence that the variability in mortality rates increases at older ages.

Stochastic mortality modelling, on the other hand, allows for mortality rates to be forecast, and thus, mortality improvements to be quantified. Lee and Carter (1992), in their seminal work, developed the Lee-Carter model which models the log-mortality rate of a certain population,

$\log(\mu_{x,t})$:

$$\log(\mu_{x,t}) = \alpha_x + \beta_x \kappa_t + \epsilon_{x,t}. \tag{3.2}$$

The α_x term represents the effect of age on the mortality rate, the β_x term represents the effect of improvement in mortality across different ages, and the κ_t term represents the aggregate improvement in mortality across time.

A more recent development is the Cairns-Blake-Dowd (CBD) model (Cairns et al., 2006), which is specifically designed at modelling older ages:

$$\operatorname{logit}(q_{x,t}) = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \overline{x}) + \epsilon_{x,t}, \tag{3.3}$$

where $\operatorname{logit} q_{x,t} = \frac{q_{x,t}}{1-q_{x,t}}$, and $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the mean age in the sample range (n is the number of ages fitted to the given population).

Additional stochastic models, including the Age Period Cohort model (presented in Denuit et al. (2011)) can be found in Cairns et al. (2009), who also offers a comparison of these models.

The κ_t term in Equations (3.2) and (3.3) can be forecasted through time series techniques to model secular improvement in mortality. A common assumption is to use a random walk with drift (see Denuit et al., 2011, and other scholarly literature). Therefore,

$$\kappa_t = \kappa_{t-1} + \theta + \xi_t, \tag{3.4}$$

where θ is the drift parameter, and ξ_t are error terms which are assumed to be independently and identically distributed (i.i.d.) normal with zero mean and variance σ^2. By considering the forecast error in κ_t, the systematic mortality risk can be estimated.

Equations (3.2) and (3.3) can be shown to be special cases of the generalised age-period-cohort (GAPC) model family (Villegas et al., 2018). These models can then fitted using maximum likelihood estimation and forecasted.

For tractability, we use the stochastic Lee-Carter model to estimate mortality rates, which we forecast using a random walk with drift (Equations (3.2) and (3.4)). We will briefly outline the methodology for this procedure.

Villegas et al. (2018) suggest first modelling the random component as a Poisson distribution of deaths:

$$D_{x,t} \sim \text{Poisson}(E^c_{x,t}\mu_{x,t}),$$

where $D_{x,t}$ represents the number of deaths in a given population, aged x, between time $t-1$ and t, $E^c_{x,t}$ represents the corresponding central exposed-to-risk, and $\mu_{x,t}$ represents the corresponding force of mortality. Then the systematic component is defined as:

$$\eta_{x,t} = \alpha_x + \beta_x \kappa_t.$$

We use the log-link, which is the canonical link function:

$$\log\left(E\left(\frac{D_{x,t}}{E^c_{x,t}}\right)\right) = \log(\mu_{x,t}) = \eta_{x,t}.$$

We also need to apply a set of parameter constraints, for details on this, more information can be found in Villegas et al. (2018). Since we have identified the Lee-Carter (and other models) as generalised non-linear models, these can be fit using standard statistical routines in `R`. Equation (3.4) can be forecasted and simulated using time series techniques.

A further issue with mortality models is their behaviour at very old ages. In designing a retirement income product, the sustainability of the payments at very old ages, even up to the maximum lifespan, must be considered. It is difficult to reliably estimate parameters for stochastic mortality models, in ages 90 and above, due to a lack of data, particularly in a small population such as Australia. Therefore, there have been various extrapolation methods developed to account for this. One such method is logistic extrapolation (Thatcher, 1999), which extrapolates the force of mortality μ_x according to:

$$\mu_x = \frac{\delta \alpha e^{\beta x}}{1 + \alpha e^{\beta x}} + \gamma. \tag{3.5}$$

If $\delta = 1$ and $\gamma = 0$, this is essentially a logistic regression of the force of mortality μ_x on the age x, and is extremely tractable. This method is suggested in Pitacco et al. (2009), who also explore alternative models such as the Weibull and Gompertz.

So far, the stochastic mortality models presented have been in discrete time. Continuous-time stochastic mortality models have also been used, which include the class of affine mortality models. One example utilised in Hanewald et al. (2013) is the model developed in Wills and Sherris (2008). This has the advantage of being easily integrated with various financial return models, which are often in continuous time.

We now turn to the incorporation of idiosyncratic risk into the stochastic mortality models described above. We consider the survival of a finite number of individuals using the binomial assumption of deaths, using an initial pool size L_x as an initial condition. This approach is widely used (see Milevsky and Salisbury, 2015, and other scholarly literature). The (stochastic) number of lives at time t, L_{x+t}, is given by a binomial distribution:

$$\text{Bin}(L_{x+t-1}, p_{x+t-1}). \tag{3.6}$$

Finally, mortality rates with systematic and idiosyncratic risk can be generated by using the output from any mortality model or simulation as an input to Equation (3.6).

The mortality models can be generalised to take into account future entrances into the pool through a population model. This will be the subject of future work.

3.2.2 Financial modelling

In its simplest form, a financial market can be comprised of a risky asset $\mathcal{S}_t$, which is invested in equities, and a risk-free asset $\mathcal{B}_t$, which is invested in cash. The risky asset $\mathcal{S}_t$ within a lifetime consumption model can be modelled using geometric Brownian motion (GBM) (Merton, 1971):

$$\frac{d\mathcal{S}_t}{\mathcal{S}_t} = \mu dt + \sigma dZ_t, \tag{3.7}$$

where Z_t is a standard Wiener process, μ is the annual expected return of the stock and σ is the annual volatility of the stock.

However, stocks often exhibit large movements which cannot be explained under a simple, constant-variance log-normal distribution as posited under geometric Brownian motion. This has led to the development of several other models which aim to quantify this unexplained volatility. The jump diffusion model incorporates an additional jump term to Equation (3.7), where the jump follows an i.i.d. Poisson process (Merton, 1976). The Heston model (Heston, 1993) takes into account stochastic volatility by adding an extra term to the Wiener process in Equation (3.7):

$$\frac{d\mathcal{S}_t}{\mathcal{S}_t} = \mu dt + \sigma \sqrt{\nu_t} dZ_t. \tag{3.8}$$

The instantaneous volatility ν_t, which allows the volatility of the stock to change over time, is modelled using a Cox-Ingorsoll-Ross (CIR) process. The volatility of the stock and the CIR process are correlated. Alternatively, more complex models are available, such as an economic scenario generator, which simulates interest rates, inflation and equity returns, presented in Hanewald et al. (2013).

The risk-free asset $\mathcal{B}_t$ can be modelled using the following differential equation:

$$\frac{d\mathcal{B}_t}{\mathcal{B}_t} = r dt,$$

where $r_t = r$ is the constant risk-free interest rate.

We choose to use the standard GBM model to simulate stock prices because of its tractability. It is also used as a standard benchmark to simulate financial risk in the retirement income product literature (see Stamos, 2008, Donnelly et al., 2013, and other scholarly literature).

The GBM model fitting procedure is outlined in Appendix A.

3.3 Modelling framework

3.3.1 The fund equation

Pitacco et al. (2009) present the basic structure of a life annuity from the perspective of a provider, who needs to hold a reserve to manage the longevity and financial risks inherent in the contract. The evolution of the reserve is first separated into three terms: the *mortality credits* Θ_t, the *financial returns* R_t and the *payout structure* b_t. The mortality credits refer to the gains from risk pooling, as those who die pass their wealth to survivors. Pitacco et al. (2009) refer to this risk-sharing mechanism as *mutuality.* The financial returns, which are passed on to the policyholder, can be deterministic or stochastic. The payout structure is developed as a function of the reserve. The reserve for an individual increases each period due to the mortality credits and financial returns, and decreases due to the benefits paid to the policyholder, as stated in the following fund equation and diagram (Figure 3.2):

$$F_t = F_{t-1}(1+\Theta_t)(1+R_t) - b_t, \quad F_0 = S - b_0, \tag{3.9}$$

using the notation defined in Section 3.1.

Pitacco et al. (2009) derive expressions for the *mortality credits* Θ_t, the *financial returns* R_t and the *payout structure* b_t in the case of a life annuity. As the life annuity pays a constant amount, the financial return R_t is a constant r. The mortality credit Θ_t increases according to the death probability $q_{x,t}$. Intuitively, this is because more people die at older ages, and so the redistribution from deceased to survivors is greater at older ages. The amount of mortality credit is also known in advance as it is calculated based on an assumed mortality basis. The payout structure b_t is written in terms of the fund value just before the benefit is paid, F_t^-. For a life annuity, this is the inverse of the annuity factor, which is priced according to the aforementioned financial and mortality basis, ensuring fairness. The formulation in terms of F_t^- is to ensure consistency with products where the drawdown strategy is determined based on the fund value, or account balance.

We extend this framework to other products. In particular, we consider the case where financial returns are no longer deterministic, and the mortality credits are no longer fixed, but reflect

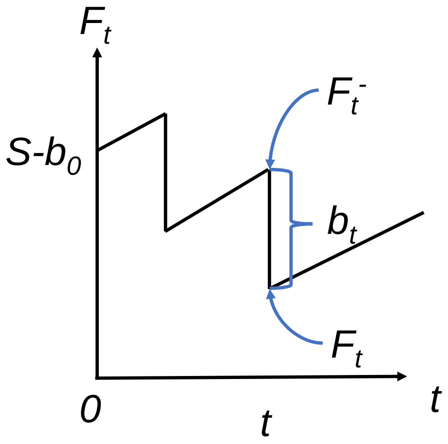

Figure 3.2: Evolution of the fund for a representative policyholder

risk pooling or risk sharing between the participant and the provider. Recall that risk pooling is the situation where a group of participants enter into an arrangement where their benefit payment, contingent on survival, depends on the mortality experience of these participants. The payout structure is independent of the financial returns and mortality credits, and can be varied by the product designer. For simplicity, for products other than the account-based pension, we only consider products which give an annuity-like payout. We summarise the results of our derivations in Table 3.1. The relevant proofs can be found in Appendix B.1.

As an example, consider the case of the group self annuitisation. Here, both financial and longevity risk are borne by the pool. Hence the financial return is a stochastic R_t, while the mortality credits depend on the survivorship of the pool, which is encapsulated in the L_{x+t} terms. Since the contract is set up to give an annuity-like payout, we note the same inverse of the annuity factor in the formulation of the payout structure.

To demonstrate the flexibility of the fund equation, we also introduce a new product design which combines features of both the longevity-indexed life annuity and mortality-linked fund. In this design, the policyholders receive mortality credits according to a longevity-indexed life annuity, while the fund is invested in risky assets. This will be called the *longevity-indexed fund* (LIF).

In addition, we have derived the representation of various features which could be added to the retirement income products introduced in Table 3.1. For simplicity, the features, namely, deferment, non-constant payments and capital guarantees are incorporated with a life annuity as the underlying product. The results are shown in Appendix B.2.

Table 3.1: Summary of guarantee structure and resultant elements of fund equation

Product	Financial risk	Longevity risk Idiosyncratic	 Systematic	R_t	$\Theta_{x,t}$	b_t
Life annuity	Provider	Provider	Provider	r_t	$\frac{l_{x+t-1}-l_{x+t}}{l_{x+t}}$	$\frac{F_{t-}}{\ddot{a}_{x+t}}$
Longevity-indexed life annuity	Provider	Provider	Individual	r_t	$\frac{l^{ref}_{x+t-1}-l^{ref}_{x+t}}{l^{ref}_{x+t}}$	$\frac{F_{t-}}{\ddot{a}^{ref}_{x+t}}$
Tontine	Provider	Provider	Pool	r_t	$\frac{L_{x+t-1}-L_{x+t}}{L_{x+t}}$	$\frac{F_{t-}}{\ddot{a}_{x+t}}$
Mortality-linked fund	Individual	Provider	Provider	R_t	$\frac{l_{x+t-1}-l_{x+t}}{l_{x+t}}$	$\frac{F_{t-}}{\ddot{a}_{x+t}}$
Longevity-indexed fund	Individual	Provider	Individual	R_t	$\frac{l^{ref}_{x+t-1}-l^{ref}_{x+t}}{l^{ref}_{x+t}}$	$\frac{F_{t-}}{\ddot{a}^{ref}_{x+t}}$
Group self annuitisation	Pool	Pool	Pool	R_t	$\frac{L_{x+t-1}-L_{x+t}}{L_{x+t}}$	$\frac{F_{t-}}{\ddot{a}_{x+t}}$
Account-based pension	Individual	Individual	Individual	R_t	0	$F_{t-}\gamma_t$

So far, the fund equation has been presented on an actuarially fair basis, which does not take into account the cost of providing the various guarantees in the products. Therefore, we will now explore one method to determine the required loading for each product based on their riskiness.

3.3.2 Capital calculations

To adequately determine the required loading, it is first necessary to calculate the capital required. The amount of capital is an indicator of the riskiness of the guarantees from the provider's perspective. The price of the product will then reflect the amount of capital loading that is applied.

First, a procedure is outlined to give a distribution of the possible required capital amounts under stochastic mortality and financial returns. Then, various risk measures are shown which indicate the required amount of capital to be held by the insurer.

To calculate a distribution of capital requirements C, the fund equation or reserve introduced in Section 3.3 needs to be adapted to reflect the actual inflows and outflows of the insurer's fund. For simplicity, we assume the product is in run-off – that is, that there are no new policies sold after the initial time. This is to ensure the amount of capital required over the lifetime of the product can be modelled adequately at inception.

The insurer's fund can be stated as:

$$L_{x+t}F^I_t = L_{x+t-1}F^I_{t-1}(1+R^I_t) - b^A_t - I^A_t, \quad L_xF^I_0 = L_xS + L_xC_0 - b^A_0, \tag{3.10}$$

where F^I_t denotes the value of the insurer's fund at time t, R^I_t denotes the value of the insurer's investment earnings from time $t-1$ to t, b_t^A is defined to be the total value of actual survival benefits paid out to the pool at time t: $L_{x+t}b_t$, and I_t^A is defined to be the total value of actual benefits, which are due to deaths occurring between time $t-1$ and t, paid at time t: $(L_{x+t-1} - L_{x+t})I_t$.

Initially, the insurer invests the initial investment S which is contributed by the policyholder, plus a certain capital held at time 0, C_0. The initial investment S is the same as that in Equation (3.9). The insurer's fund value F_t^I accumulates due to the actual financial returns earned by the insurer R_t^I and decreases due to the actual survival benefits paid to the pool, b_t^A and actual death benefits paid to the pool, I_t^A.

Since b_t^A and R_t^I are stochastic, Equation (3.10) is stochastic. Let C_0 be the random initial capital required such that $F_{\omega-x}^I = 0$. Given a large number of simulations, we can calculate an empirical distribution for C_0.

Given this distribution, a mapping from this distribution to a real number must be made which corresponds to the hypothetical amount of capital held by the company. This is the role of a risk measure. Arguably the most common risk measure is the Value-at-Risk (VaR). Here, we define the VaR for the capital, C_0^*, as the smallest amount of capital at time 0, C_0 such that the probability that the terminal value of the fund $F_{\omega-x}$ is greater than 0 is α (McNeil et al., 2015):

$$C_0^* = \mathrm{VaR}_\alpha(C_0) = \inf\{C_0 \in \mathbb{R} : Pr(F_{\omega-x} \geq 0) = \alpha\}$$

Typical values for α are 0.95 or 0.99.

Expected shortfall (ES) is another risk measure which we define as the average amount of initial capital C_0 exceeding a certain VaR $_\alpha$, $ES = E(C_0|C_0 \geq \mathrm{VaR}_\alpha)$. It seeks to address shortcomings in the VaR by considering all possible extreme losses which could impact the insurer. In particular, it can be shown that using heavy-tailed distributions, the difference between ES and VaR can be quite substantial (McNeil et al., 2015), as the losses in the tails are not adequately considered by VaR.

The theoretical VaR and ES can be approximated in the case of empirical distributions through L-estimators, which are linear estimates of order statistics McNeil et al. (2015).

For our purposes, we determine the initial capital requirement using the Value at Risk at a confidence level $\alpha = 0.99$: $C_0^* = \mathrm{VaR}_{0.99}\{C_0\}$.

3.3.3 Pricing

Given an initial capital requirement C_0^*, we now outline an approach to calculate the price charged on the initial investment S. We use the risk margin RM as defined under Solvency II (EIOPA, 2014):

$$\text{RM} = \text{CoC} \sum_{t=0}^{\omega - x} \frac{C_t^*}{(1+r_f)^t},$$

where CoC denotes the cost of capital, C_t^* denotes the capital required at time t, and r_f denotes the risk-free rate. Intuitively, this means that the provider of capital receives an amount equal to $\text{CoC} \times C_t^*$ at each time t. The present value of this at the risk-free rate is the risk margin RM.

The capital required, C_t^*, at future times $t = 1, \ldots \omega - x$, is calculated as a constant proportion ζ of the expected fund value $E[F_t]$. This proportion is calculated with reference to the price charged at time 1:

$$C_0^* = \zeta \times E[F_1] \tag{3.11}$$

$$\zeta = \frac{C_0^*}{E[F_1]} \tag{3.12}$$

$$C_t^* = \zeta E[F_t] \tag{3.13}$$

We first add the policyholder's initial investment S to the risk margin RM to give the loaded investment S^*. Then we express the risk margin as a percentage p of the initial investment:

$$S^* = S + \text{RM}$$

$$S^* = S + pS \tag{3.14}$$

To allow a fair comparison between products, we set the loaded investment S^* to be the same across all investments. The equivalent unloaded investment S can be found by rearranging Equation (3.14): $S = \frac{S^*}{1+p}$. This formulation implies that the riskier the product, the more the benefits will be reduced, for a given purchase price.

3.4 Evaluation framework

We now evaluate the retirement income products using two main dimensions: quantitative measures of value and risk, and utility analysis. The specification will be informed by a brief analysis of behavioural features. These measures will be used to communicate outcomes to academics, industry and policymakers.

3.4.1 Behavioural features

Studies of people's preferences, particularly as it relates to consumption in retirement, have been widely undertaken in the behavioural economics literature. As stated in the introduction, the decision on purchasing a retirement income product is a complex one, requiring retirees to solve a "risky, long-horizon, multi-dimensional problem" (Iskhakov et al., 2015). The complexity inherent in products with longevity guarantees and options only complicates this decision. Bateman et al. (2018) consider individuals who are presented with the features of a phased withdrawal account and life annuity. Through experiments, the authors show that the protection against longevity and financial risk provided by the life annuity is more valued, when financially literate individuals are able to acquire information about the products at the time of purchase. This knowledge is termed just-in-time knowledge, and implies that retirement income products which incorporate financial and longevity protection features should not be unnecessarily complicated, to allow retirees to appreciate their worth. In terms of the payout of the products, they should be rising in real terms rather than falling. This is shown in survey work performed in Beshears et al. (2014), where 50% of the people surveyed preferred an increasing stream of payments in real terms, as opposed to only 19% who preferred a decreasing stream. This is in contrast to empirical evidence where retirees' consumption habits lessen as they age (Beshears et al., 2014).

Qualitative measures such as surveys are widely used in industry to evaluate retirement income products, of which the National Seniors survey (National Seniors and Challenger, 2018) is one prominent example. The most important financial goals in retirement include: regular, constant income, longevity protection, protection against aged care costs, inflation protection and capacity to withdraw savings. The authors also consider the importance of the bequest motive, and found that most retirees did not consider it to be a priority when setting their goals in retirement. Of those who did consider it important to leave a bequest, only about 10% of those would minimise current consumption in order to achieve a bequest.

However, it is important to note that preferences elicited from experiments, as in the case of Beshears et al. (2014), may differ from the preferences found in survey work (for example, National Seniors and Challenger, 2018). This may be a consequence of the experimental design or framing effects. Thus, qualitative findings should not be interpreted as providing a definitive answer on the features most desirable for retirees in the selection of products. Nevertheless, survey work can offer guidance on which features can be quantified in the utility analysis. This will be elaborated upon in Subsection 4.6.3.

3.4.2 Quantitative measures

Quantitative analysis offers a contrasting approach to the preceding analysis of individuals' preferences. Rather than empirically examining the demand for products and their features, quantitative measures take the product characteristics as given, then aim to quantify the value and riskiness of such products.

3.4.2.1 Actuarial present value

Firstly, the commonly-used notion of actuarial present value (APV) is central to the analysis of retirement income products. The APV takes the benefit at each time t, conditional on survival at time t, and discounts it to time 0. In theory:

$$APV = \sum_{t=0}^{\omega-x} {}_tp_x v^t b_t.$$

The discount rate embedded in v^t is the best estimate of the risk-free rate; the probability of survival ${}_tp_x$ used in the APV calculation is a best estimate of the survival probability and the benefit b_t is given by the product provider. Mitchell et al. (1999) use this basic notion in the context of life annuities and extend it to make it more realistic. Firstly, they use the term structure of interest rates on Treasury bonds to calculate expected short rates in future time periods. They then calculate the spread between BAA-rated corporate bonds and 10 year Treasury bonds, assuming it is constant for all maturities. The spread is then added to the expected future short rates to give expected future corporate bond yields, which they use for the discount factor, in addition to using the risk-free rate. Secondly, the authors use a cohort life table, which takes into account mortality improvements, to determine the mortality rates. The data for the life tables are sourced from both a population life table and an annuitant life table. The latter takes into account adverse selection among annuitants. Thirdly, the effect of US taxes is taken into account. Therefore, a range actuarial present values are found, depending on the mortality and interest rate basis used.

The APV of the life annuity is not expected to be the same as the initial capital invested, S. This is because the interest rates and mortality rates used by the annuity provider to calculate S and b_t could be quite different from best estimate values. The interest rate could be lower than the risk-free rate and mortality rates could be more conservative, making b_t lower than it would be under a fair basis. This is to ensure that the insurer makes a profit and is able to deliver a return on capital to its investors.

A comparison between the APV of the product and the initial investment made S can provide an indication of the value of the product. For example, if the APV is less than S it means that over the lifetime of the individual, in expectation, the product is delivering less than the price used to purchase it. This is the idea behind the *money's worth* ratio (Mitchell et al., 1999), which is defined to be $\frac{APV}{S}$. Under fair pricing, the basis used to determine b_t and the APV are the same; therefore, the money's worth is 1. If the insurer applies a loading, then the payout b_t is reduced and therefore the APV is similarly lowered, reducing the money's worth.

In the context of simulated retirement income products, this definition is adapted slightly. The discount factor v^t is a constant risk-free rate, the survival probability ${}_tp_x$ is the simulated survival probability for a hypothetical infinitely large pool, and the benefit b_t is the corresponding simulated benefit. We then construct the APV in percentage terms, similar to the money's worth, using a large number of simulated survival probabilities to construct a *distribution.* We refer to this distribution as the APV in percentage terms (APV%):

$$APV\% = \frac{APV_i}{S},$$

where APV_i is defined as the actuarial present value calculated under each simulated mortality scenario.

The expected APV is calculated by taking the expectation over all mortality and financial scenarios, consistent with Hanewald et al. (2013). This formulation gives the present value of the amount an individual, as part of a large diversified pool, would expect to receive. We construct the money's worth using the *expected* APV, and hence it is distinct from the APV% defined earlier. The money's worth is defined as:

$$\text{Money's Worth} = \frac{\frac{1}{N}\sum_{i=1}^{N} APV_i}{S},$$

where APV_i is defined as above, and N is the number of simulations performed.

3.4.2.2 Australian Government Actuary risk measure

The standard deviation is a well-known risk measure to gauge the volatility of a series of cash flows. However the standard deviation penalises upside and downside risk equally, and also does not take anchoring consumption to a certain level into account.

The Australian Government Actuary (2018) risk measure penalises downside risk, considering only the negative deviance between the benefit at time t, b_t and the first benefit, *inflation-adjusted* to time t, denoted by $b_{0,t}$. Thus, it focuses only on downside risk, taking the anchor point to be the real value of the first payment the retiree receives. This is motivated by the concept of loss aversion, where losses and gains are perceived in relation to a reference point (Tversky and Kahneman, 1991) . A measure of downside risk with reference to the initial payment thus mimics the natural aversion to losses compared to gains.

The risk measure is calculated similar to a coefficient of variation: $\frac{\sigma}{b_{0,t}}$, where σ is given by:

$$\sigma = \sqrt{\frac{1}{\omega - x}\sum_{t=0}^{\omega - x} \max(b_{0,t} - b_t, 0)^2}. \tag{3.15}$$

Furthermore, if the benefit has reached zero, the corresponding deviation is not counted in the risk measure. In this context, this means that all pool members have died and the pool is no longer operating. Although the measure is defined based on payments in real terms, we assume for simplicity that it applies equally to payments in nominal terms, since we do not incorporate a model for inflation.

In the simulation context, the AGA risk measure is calculated as an average over all simulations, similar to the expected APV:

$$\text{AGA Risk Measure} = \frac{\frac{1}{N}\sum_{i=1}^{N} AGA_i}{S},$$

where APV_i is defined as the AGA calculated under each simulated series of benefits b_t, and N is the number of simulations performed.

Furthermore, we do not adopt the Australian Government Actuary's proposal to truncate this risk measure at age 100, as in the future, mortality improvements mean that there will be a small but significant number of people living past this age.

3.4.3 Utility framework

In the academic literature, the most common way to evaluate a series of uncertain future cash flows is through a utility function coupled with a expected utility framework. This framework is more sophisticated than a risk measure because it also takes into account time preferences and survival probability, along with risk aversion. First, the simple case of constant relative risk aversion (CRRA) is presented. Then, the bequest motive and habit formation will be considered.

3.4.3.1 CRRA utility

The CRRA utility model has the property that a constant proportion of the wealth is invested in risky assets as wealth increases. In Hanewald et al. (2013) and the related academic literature, a utility function which satisfies this property is given by:

$$U(b_t) = \frac{b_t^{1-\rho} - 1}{1-\rho}, \tag{3.16}$$

where $\rho \in (0, \infty) \setminus 1$ is the relative risk aversion parameter. If $\rho > 1$, it means the individual is risk averse, and if $0 < \rho < 1$, the individual is risk-seeking. The risk-neutral case is the limit where $\rho \to 1$ in Equation (3.16): $U(b_t) = \log b_t$. An alternative to this, which is less frequently used, is a utility function which satisfies constant absolute risk aversion (CARA). This has the property that a constant amount of the wealth is invested in risky assets as wealth increases.

An example of such a function is the exponential utility (see, for example Bell et al., 2017):

$$U(b_t) = -\frac{1}{\varrho} e^{-\varrho b_t}, \tag{3.17}$$

where ϱ is a constant, with similar interpretation to ρ in equation (3.16).

These functions give the utility of a single benefit payment b_t. These can then be incorporated into the discounted utility framework (Hanewald et al., 2013) to evaluate outcomes over the lifetime of the individual:

$$U_0 = E_0 \left[\sum_{t=0}^{\omega - x} {}_t p_x \beta^t U(b_t) \right], \tag{3.18}$$

where β is the time preference parameter and $U(\cdot)$ represents any utility function. The survival probability ${}_t p_x$ can either be the objective survival probability, taken from a life table, or the subjective survival probability (Weinert and Gründl, 2016), measured using surveys.

Given the discounted utility framework described above, we can extend this to calculate the certainty equivalent. The certainty equivalent measures the value of a constant consumption stream whose discounted utility is equal to the discounted utility of the product. In our case, an individual would be indifferent between purchasing the product and a life annuity of an annual payment of η. Stated mathematically we find this constant η such that:

$$E_0 \left[\sum_{t=0}^{\omega - x} {}_t p_x \beta^t U(b_t) \right] = E_0 \left[\sum_{t=0}^{\omega - x} {}_t p_x \beta^t U(\eta) \right], \tag{3.19}$$

where the terms are defined as above.

Bell et al. (2017) propose the extension of the discounted utility framework to the following form, in order to incorporate a bequest motive:

$$U_0 = E_0 \left[\sum_{t=0}^{\omega - x} {}_t p_x U(b_t) + {}_{t-1|}q_x U(I_t) \left(\frac{\phi}{1 - \phi} \right)^\rho \right],$$

where $U(\cdot)$ is given by Equation (3.16), I_t denotes the amount of bequest and ϕ is the parameter for the strength of bequest motive.

3.4.3.2 Habit formation

Consumers often refer to their past consumption in deciding their present consumption levels, a phenomenon known as *habit formation*. The discounted utility framework presented thus far assumes time separable utility, and cannot take this desire into account. The earliest literature in this area expressed habit formation mathematically using a utility framework by replacing b_t with $b_t - X_t$ in Equation (3.16) (Constantinides, 1990), where X_t is the subsistence habit.

This is referred to as *additive* habit formation, and is given by the following equation:

$$U(b_t) = \frac{(b_t - X_t)^{1-\rho} - 1}{1-\rho}, \tag{3.20}$$

with the subsistence habit X_t at time t defined as:

$$X_t = e^{-At}X_0 + B\int_0^{t-1} e^{A(s-t)} b_s \, ds, \tag{3.21}$$

where A and B are constant parameters. It can be seen here that present consumption at time t can be written as a weighted average of past consumption and the initial habit X_0, with more recent periods weighted more than more distant periods. The higher the value of A, the more weight is placed on later periods of consumption in the determination of X_t. The higher the value of B, the less weight is placed on the initial habit X_0.

However, a major problem with the additive habit formation in Equation (3.20) is that under certain scenarios, this equation can give infinitely negative utility (Carroll, 2000). Practically, also this formulation is difficult to incorporate into a discrete time framework, especially when the benefit does not evolve continuously. Fuhrer and Klein (2006) have developed an alternative specification where habits evolve in a *multiplicative* fashion:

$$U(b_t) = \frac{(b_t / X_t^{\gamma})^{1-\rho} - 1}{1-\rho}, \tag{3.22}$$

with the reference habit X_t at time t defined as:

$$X_t = X_{t-1} + \lambda(b_{t-1} - X_{t-1}). \tag{3.23}$$

The reference habit X_t depends on the past reference habit as well as the past benefit. The parameters in this model can be easily interpreted. The importance of the habit in the utility function is denoted by $\gamma \in [0, 1]$. The persistence of previous habits is denoted by $\lambda \in [0, 1]$. In particular, the special case of $\lambda = 0$ deserves consideration. It can be shown that in this case, habit formation reduces to a situation of consumption smoothing according to the level of consumption last period and its change in this period (Fuhrer and Klein, 2006). As λ increases, more distant periods of past consumption also have an impact on present consumption. The case of $\gamma = 0$ is also instructive, as Equation (3.22) reduces to the simple case of CRRA utility.

Multiplicative habit formation is also presented in Davidoff et al. (2005) who show that the annuitisation decision depends on the initial habit X_0. If the initial habit is too high or too low, then the deferred consumption induced by the annuity is sub-optimal.

Iskhakov et al. (2015) analyse the case of the fixed consumption floor, or constant habit. This represents the minimum standard of living required by a retiree to live comfortably. In the

Australian context, Iskhakov et al. (2015) express this as a benchmark of minimum living standards, such as the ASFA Modest Retirement Standard (ASFA, 2018, in Iskhakov et al. (2015)). They find that lower-wealth retirees would increase their allocation to annuities, since this is the only asset which can guarantee consumption levels. To model the constant habit using the additive habit formation of Equations (3.20) and (3.21), the constants A and B need to be set to 0. All habit levels are then set to the initial habit level.

In summary, there is a wide choice of features to incorporate into a utility analysis, as well as parameters in each function to be estimated. Incorporating habit formation ensures smoothness of consumption and penalises excessive variability in income from one period to the next, a desirable feature for retirees (van Bilsen and Linders, 2019, National Seniors and Challenger, 2018). We set the relative risk aversion, time preference parameters and the constants in the utility framework at an average value found in the literature. The use of habit formation depends on the setting of an initial habit X_0. For simplicity, we assume this is the expected first benefit of a life annuity, similar to the Australian Government Actuary (2018) risk measure, where it is set to the expected first benefit of the product. It is important to note that this initial habit could be different for differing amounts of wealth, as intuitively, wealthier retirees have a higher standard of living than less wealthy retirees. The bequest motive is not incorporated since this is a lower priority for most retirees (National Seniors and Challenger, 2018), and it would unnecessarily complicate the analysis.

CHAPTER 4

RESULTS

This chapter presents the practical contribution of this thesis through simulations, building on the theoretical development of the fund equation for a wide variety of retirement income products. These simulations are conducted to elucidate the impact of changing the guarantee structure on retirees' incomes by focusing on the benefit payout structure, capital requirements and resulting prices for illustrative products.

We first define the mortality and financial environment in Subsection 4.1, which will provide a common basis for the simulations. Then, we illustrate the various financial and mortality guarantees by comparing the simulated benefits in Subsection 4.1.2. The cost of such guarantees will be quantified by calculating the capital requirements. We then price the product. Once these prices are obtained, we evaluate the products using risk-based measures and utility functions which are informed by the behavioural economics literature. Lastly, we perform sensitivity analysis in Subsection 4.7.

4.1 Central assumptions

We assume throughout this section that all products are sold to a pool size of $L_x = 1000$ males aged 60 each with an assumed initial wealth of $S = \$100,000$. The assumed mortality basis of this population is common across all products, and is the same as that of the Australian population. That is, we do not take into account adverse selection in any of the products that we model. This mortality basis will be further described in the next section.

4.1.1 Mortality assumptions

Mortality rates are modelled from ages 50 to 89 according to the Lee-Carter model (Lee and Carter, 1992) for Australian males, using data from 1967 to 2016. The data is sourced from the Human Mortality Database (Human Mortality Database, 2019). The fitted rates are then forecasted through simulating a random walk with drift. Rates for higher ages are extrapolated for each forecast using the approach of Thatcher (1999), where the logistic regression is fitted from ages 70 to 89 and used to predict the force of mortality μ_x from ages 90 to 109. We forecast the mortality rates for 51 years, in order to provide cohort mortality rates which take into account mortality improvement from age 60 to 109. These rates are converted into death probabilities $q_x = 1 - e^{-\mu_x}$. We perform 5000 simulations, which proved to be large enough to obtain stable estimates of the different evaluation metrics.

The expected value of these simulations, being the best estimate of the cohort mortality rates for the Australian male population, is set as the reference population for the longevity-indexed life annuity (LLLA) and longevity-indexed fund (LIF). Hence, the mortality credits and payments, respectively are given in terms of $l^{ref}_{x+t}, \ddot{a}^{ref}_{x+t}$ for the LLLA and LIF.

The idiosyncratic risk is simulated using a binomial distribution (see Equation (3.6)) with the above simulations as input, with initial pool size $L_x = 1000$ individuals. This basis, which takes into account the risk of the finite pool size, is used to calculate the mortality credits for the GSA and tontine, given in terms of L_{x+t}.

The expected value of the simulations, *after taking into account idiosyncratic risk*, is set as the basis for the mortality credits for the life annuity and MLF, given in terms of l_{x+t}. The payments for the remaining products apart from the account-based pension (i.e. the life annuity, tontine, MLF, and GSA), also use this as the basis for the calculation of $\ddot{a}_{x+t}$.

The details of the models can be found in Section 3.2.1.

4.1.2 Financial assumptions

Financial assumptions are defined according to the methodology outlined in Subsection 3.2.2. We use Australian All Ordinaries Accumulation Data from 1980 to 2016 to calibrate the parameters for the Geometric Brownian Motion stock model. We find that the fitted annual expected return $E[R^{\text{stock}}] = \hat{\mu} = 0.1183$ and the fitted annual volatility $\hat{\sigma} = 0.1735$. The details of the fitting procedure can be found in Appendix A.

In the context of our simulation model, we simulate 5000 stock returns, using the parameters $\mu = 0.1083$, $\sigma = 0.1735$. Note that the drift term μ has been reduced by 1%, because we take into account the investment costs faced by the provider in investing in equities. The level of the fee is similar to the fee charged in the Mercer LifetimePlus product, which was between 0.78% to 0.98% (Mercer, 2017). The risk-free asset r_f is assumed to be a constant 4% through time. This implies an equity risk premium (ERP) of 6.83%. This is quite high, and we address this by performing sensitivity analysis using a lower ERP in Subsection 4.7.6[1].

We use this estimate to create a two-asset model which is assumed to comprise of a $\alpha\%$ allocation to stocks, and $(100 - \alpha)\%$ allocation to the risk-free asset:

$$R_t = \alpha\% \times R^{\text{stock}} + (100 - \alpha)\% \times r_f \tag{4.1}$$

This model will be used with varying α to describe the financial strategies of the insurer and individual in the next section.

4.1.3 Financial strategies

We first differentiate between the products by the entity which bears the financial risk. For products where the insurer bears the financial risk, their investment portfolio is assumed to follow a strategy where $\alpha = 10$ in Equation (4.1), with an annual expected return of 4.78 % and an annual volatility of 1.73 %. We do not assume that the insurer invests their portfolio wholly in risk-free assets. The insurer cannot have a perfect matching of assets and liabilities, because of the unavailability of long-dated bonds which match the cash flow of the insurer's liabilities. In this work, we do not consider the availability or cost of hedging strategies.

For products where individuals bear the financial risk, we assume they follow a conservative investment strategy where $\alpha = 30$ in Equation (4.1). This choice of financial strategy is similar to the Mercer LifetimePlus product, where the strategic asset allocation was 35% growth and 65% defensive (Mercer, 2017). The annual expected return is 6.35% and annual volatility is 5.2%.

The effect of the choice of investment strategies will be discussed in later sections.

For the account-based pension (ABP), a type of phased withdrawal, the drawdown rates are those specified in Table 4.1, which are the minimum drawdown rates mandated by the Australian Government.

4.2 Comparison of payouts

As noted in the Literature Review in Section 2, retirement income products can be grouped into products where the provider guarantees longevity risk, the provider shares longevity risk

[1]We find in this subsection that the overall comparison and ranking of the products do not change, but the insurer's cost of the financial guarantee is greatly increased

Table 4.1: Minimum drawdown rates for the account-based pension

Age	Annual payment as a % of account balance
55-64	4%
65-74	5%
75-79	6%
80-84	7%
85-89	9%
90-94	11%
95+	14%

Table 4.2: Risk-sharing in retirement income products

Product	Financial risk	Longevity risk	
		Idiosyncratic	Systematic
Life annuity	Provider	Provider	Provider
Longevity-indexed life annuity	Provider	Provider	Individual
Tontine	Provider	Pool	Pool
Mortality-linked fund	Individual	Provider	Provider
Longevity-indexed fund	Individual	Provider	Individual
Group self annuitisation	Pool	Pool	Pool
Account-based pension	Individual	Individual	Individual

with the individual, and the pool of policyholders shares longevity risk among themselves. This last case is where the provider does not provide any guarantee of longevity risk. The financial risk can be borne either by the provider or individual. The case of the Group Self Annuitisation (GSA) is such that all risks are pooled. This implies that the financial strategy must also be common among all individuals in the pool. This is because if individuals could choose their financial strategy, their share of the pool could grow to an inequitable size relative to the other participants (Sabin, 2010). The products and their corresponding guarantee structure is summarised in Table 4.2.

We now consider the effect of the guarantee structure on the benefit payments contingent on survival, shown in Figure 4.1. A life annuity gives a flat payment of $6173.65 as long as the policyholder is alive. For the first 20 years, the tontine and LLLA behave like a life annuity in expectation. This is because the payout rules for these products stipulate a constant, annuity-like payout in expectation. At very old ages, the payments for the LLLA are more variable. This reflects the uncertainty in forecasting mortality at older ages, as in this product the only source of variability is uncertainty in future mortality at older ages in the population as a whole.

At very old ages, the payments for the GSA and tontine are extremely variable. Indeed, the

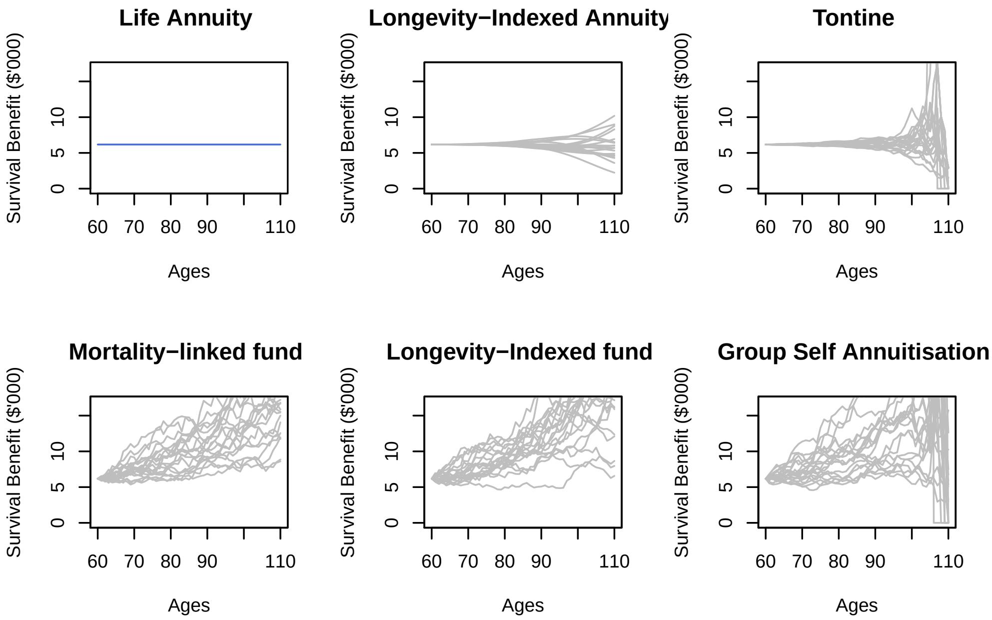

Figure 4.1: 20 simulations of survival benefits for retirement income products

5th and 95th percentile for the benefit payments of the GSA are \$6711.93 and \$18674.95 respectively. For the tontine, the corresponding payments are \$5229.16 and \$6881.27. This is due to two factors: the small number of people remaining in the pool at very old ages; and the 'last survivor' condition (Milevsky and Salisbury, 2015). This condition states that the last survivor in the pool receives whatever is remaining in the fund. This amount could be several times the payment in a typical year near the beginning of the contract, especially if mortality rates have been higher than expected. The payouts for the GSA are more variable than those of a tontine due to the addition of financial risk from the policyholder's perspective. The pool-specific characteristics described here are removed from the product design in the LLLA, LIF and MLF; in particular, the LLLA (LIF) can be thought of as a smoothed tontine (GSA) applying to a pool size which approaches infinity.

For a MLF, LIF and GSA, the payments slope upwards in expectation. This is because the benefits depend on a fixed annuity factor (see Table 3.1), which is priced so as to give a constant payment, assuming the interest rate is 4%. However, the expected financial returns which are passed on the policyholder are higher, at 6.35% (see Equation (4.1)). This difference leads to a lower benefit being paid initially, so that benefits grow over time with an increasing fund value. This also highlights the value of participating contracts to policyholders – they can earn a higher payout in expectation, albeit with higher volatility.

Figure 4.2 shows the corresponding simulations for the account-based pension (ABP), a type of phased withdrawal product. As the policyholder is able to keep their amount invested

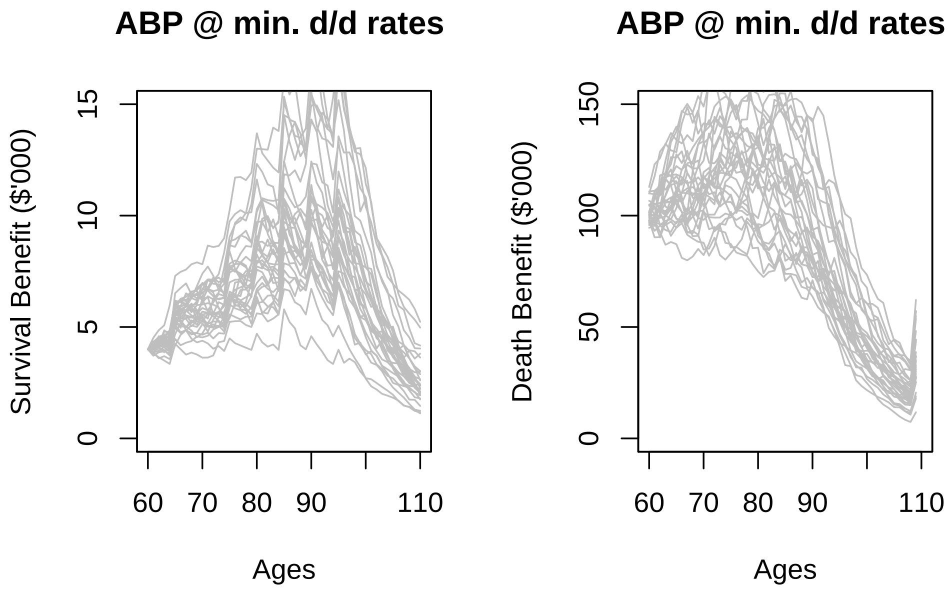

Figure 4.2: 20 simulations for the account-based pension

if they die, we differentiate between the benefits contingent on survival (left) and benefits contingent on death during a particular year (right). The large payment in the death benefit at the terminal age 110 reflects the return of the fund at the end of the contract, when all individuals are assumed to have died.

4.3 Comparison of riskiness

The capital distribution C for these products are shown in Figure 4.3, with the horizontal line in the plot representing the estimated quantile at 50% (the median) and the number at the right hand side representing the 99th percentile. The 50th (99th) percentile is the amount of capital required to be held such that the insurer is solvent until the terminal time in 50% (99%) of cases. We can see that the median of the capital distribution for all products with financial guarantees (annuity, tontine and LLLA) is below zero. This is because, on average, the insurer is expected to profit from this contract, as the expected return is higher than the promised financial return to the policyholder (Equation (4.1) with $\alpha = 10$). For instance, the life annuity is expected to make a profit of approximately 7.5% for each dollar contributed by the policyholder over the lifetime of the contract. The median of the capital distributions for the products without financial guarantees (MLF, LIF, GSA and ABP) are approximately zero, as the mortality risk is accounted for in expectation in the pricing formulae of the contracts. The capital for the GSA is in some cases below zero, since the last survivor condition implies the contract is terminated early and the provider keeps the remaining money in the fund. A similar logic applies for the ABP where the policyholder keeps what is remaining in the fund

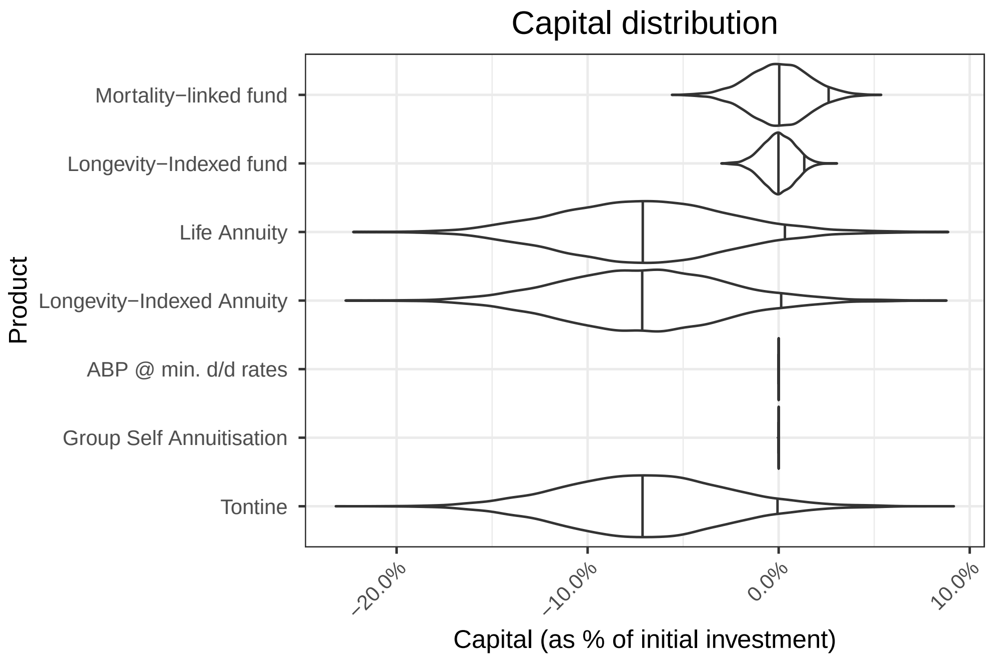

Figure 4.3: Comparison of the capital distribution for retirement income products

upon death.

The 99th percentile of C, displayed as a text label in Figure 4.3, offers a more realistic picture of riskiness in each product. For instance, the capital required to be held for the life annuity at the 99th percentile over the lifetime of the contract equates to 3.73%. We can see that the financial guarantee causes a large increase in the required capital, due to the high volatility. The difference in the mortality guarantees do not translate to a substantial effect on the capital required for products with financial guarantees. Nevertheless, for such products (namely, annuity, tontine and LLLA), the greater the mortality guarantees, the greater the amount of capital required. Hence, we can see that the riskiness of the longevity-indexed life annuity is between the life annuity and tontine. The riskiness of the stand-alone mortality guarantee can be seen in the MLF, and this can be compared with the riskiness of guaranteeing only the idiosyncratic mortality risk in the LIF.

Another important feature of Figure 4.3 is the shape of the distribution. In particular, the capital distribution for the products with financial guarantees (annuity, tontine and LLLA) is far wider than the products without such guarantees. This implies that the provider could receive a large windfall if financial markets perform, as well as, or better than expected. This is balanced by the fact that a sum of capital still needs to be held in case of extreme market scenarios. The numerical differences in the capital requirements are presented in Table 4.3, which shows the capital distribution at various quantiles and compares it with the guarantee structure of each product.

Table 4.3: Distribution of capital for each product: S = Systematic mortality risk, I = idiosyncratic mortality risk, F = financial risk

Product	Guarantee	5%	25%	Median	75%	95%	97.5%	99%
Life Annuity	SIF	-0.1422	-0.1005	-0.0710	-0.0415	0.0027	0.0164	0.0373
MLF	SI-	-0.0262	-0.0103	0.0003	0.0107	0.0262	0.0309	0.0359
LLLA	-IF	-0.1400	-0.0998	-0.0710	-0.0423	0.0007	0.0146	0.0295
Tontine	–F	-0.1399	-0.0991	-0.0708	-0.0432	-0.0003	0.0118	0.0271
LIF	-I-	-0.0142	-0.0057	0.0000	0.0057	0.0131	0.0157	0.0184
ABP	—	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
GSA	—	-0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 4.4: Prices of products as proportion of initial investment

Product	Price (%)
Life annuity	5.13
LLLA	4.07
Tontine	3.74
MLF	6.27
LIF	3.22
GSA	0
ABP	0

4.4 Comparison of prices

We show the corresponding prices of the products in Table 4.4. We assume there is no provision for expenses or other costs. The product with the highest price is the MLF, not the life annuity. This apparent contradiction is explained by considering the formula used to calculate the capital at each time t (Equation (3.11)). Since the amount of capital at future times $C_t^*,\ t = 1, \ldots, \omega - x$ are a constant percentage of the fund value, a greater fund value means that the value of mortality credits needs to be higher, resulting in a higher price charged to the policyholder. Intuitively, this can be explained as an interaction between mortality and financial risk. If the product allows the policyholder to participate in equity returns, ceteris paribus, the mortality credits are more valuable, since they are being paid on a higher fund value. Therefore, a higher price should be charged, even though the product is less risky from the provider's perspective.

The price for the GSA is set to zero, since we assume the provider keeps any surplus capital.

We now turn to the evaluation of products after taking into account loadings. This enables us to highlight the value of each product from the policyholder's perspective, as if they were going to buy such products. First we present the distribution of benefit payments, then several risk and utility measures.

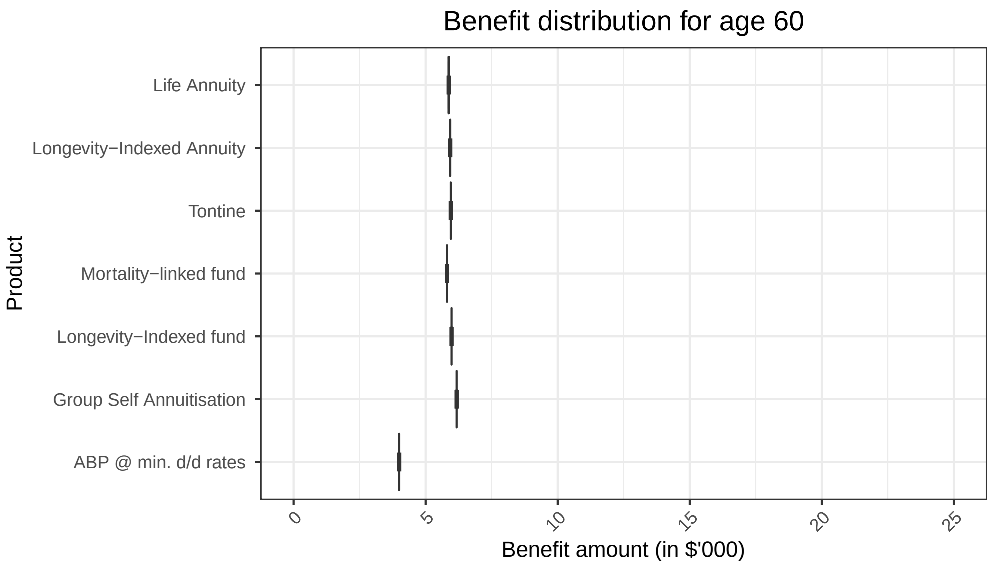

Figure 4.4: Benefits at age 60

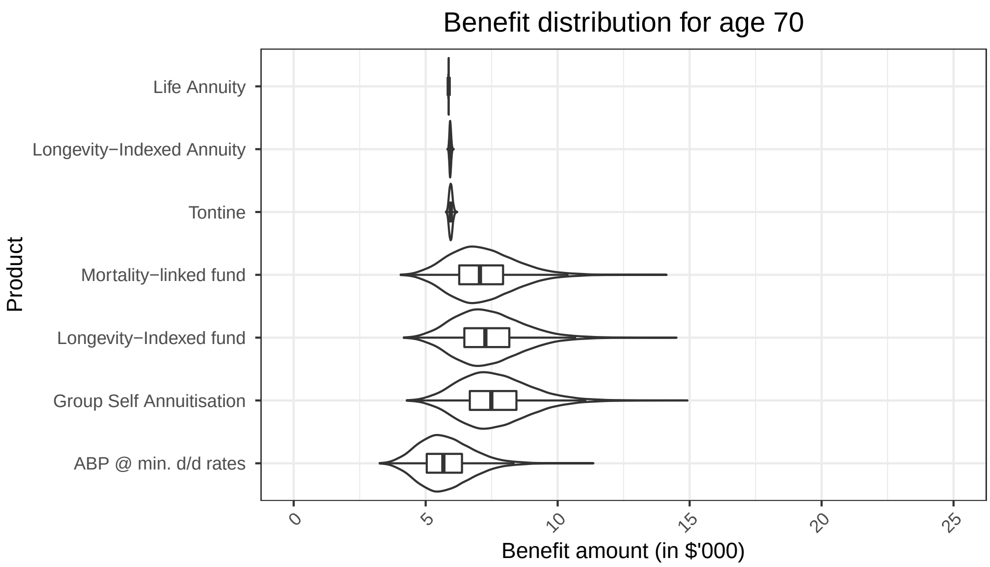

Figure 4.5: Benefits at age 70

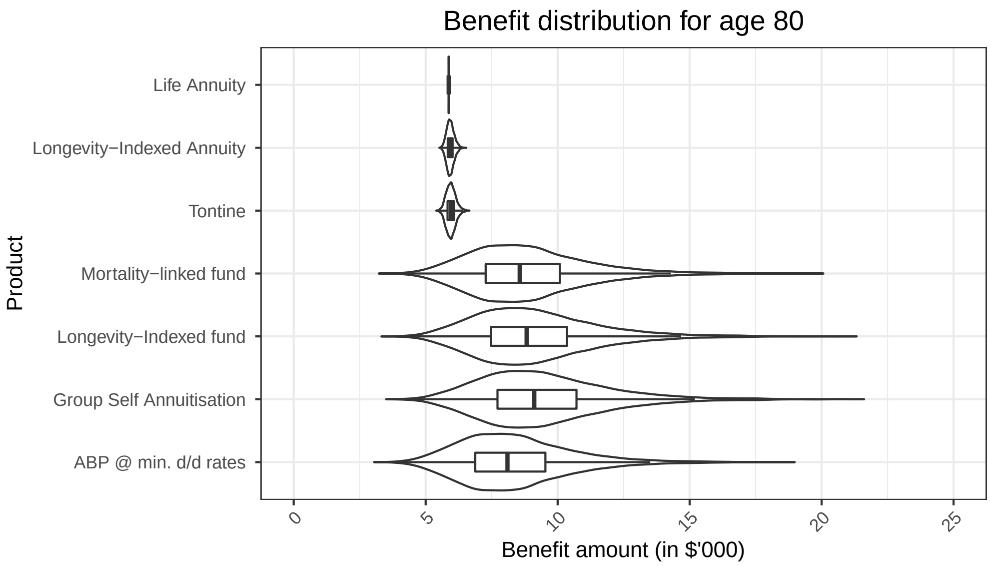

Figure 4.6: Benefits at age 80

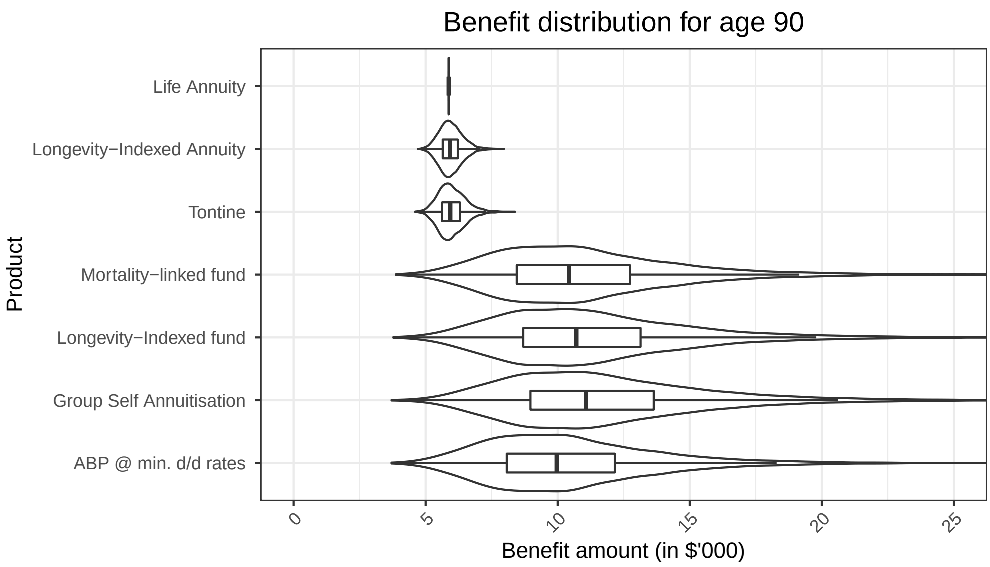

Figure 4.7: Benefits at age 90

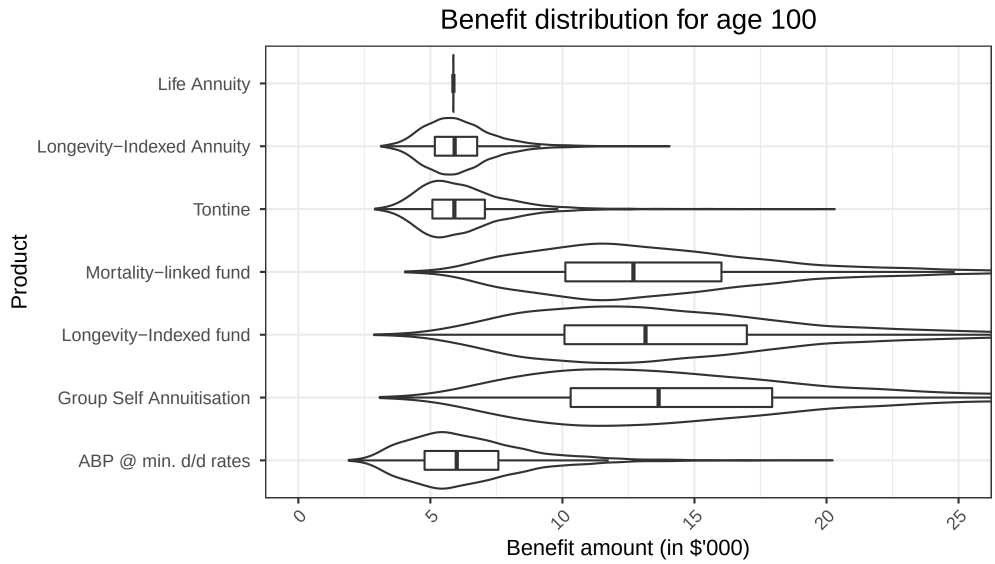

Figure 4.8: Benefits at age 100

4.5 Comparison of benefits

The distribution of benefits for selected ages is presented in Figures 4.4 to 4.8.

At age 60, all products pay out a deterministic value. The ABP pays the lowest amount of \$4,000, which corresponds to a 4% drawdown. The next lowest value is the MLF, which has the highest price. The GSA pays the highest benefit since the policyholder does not pay a loading.

At age 70, the distribution is clearly able to be separated into products according to the bearer of the financial risk. Products with equity participation exhibit a higher volatility of payments, and higher median payment, compared to those without. The boxplot also shows that the 25th percentile of payments in the MLF is comparable to the life annuity payment, which demonstrates that adding a reasonable amount of equity participation (30%), ceteris paribus, can lead to higher outcomes in the majority of cases. Similar arguments can be made by comparing the LLLA and LIF, and tontine and GSA.

At ages 80 and 90, the overall shape of payments is similar to that at 70, however the distribution of payments for contracts with equity participation is progressively shifted further to the right. The distributions of the LLLA and tontine also widen, indicating greater uncertainty in predicting future mortality improvements.

At age 100, the overall shape of the LLLA and tontine change markedly, with the right tail becoming quite pronounced. This is because of the high uncertainty in the mortality forecast. This high uncertainty, which is passed onto retirees, is likely to be undesirable for them,

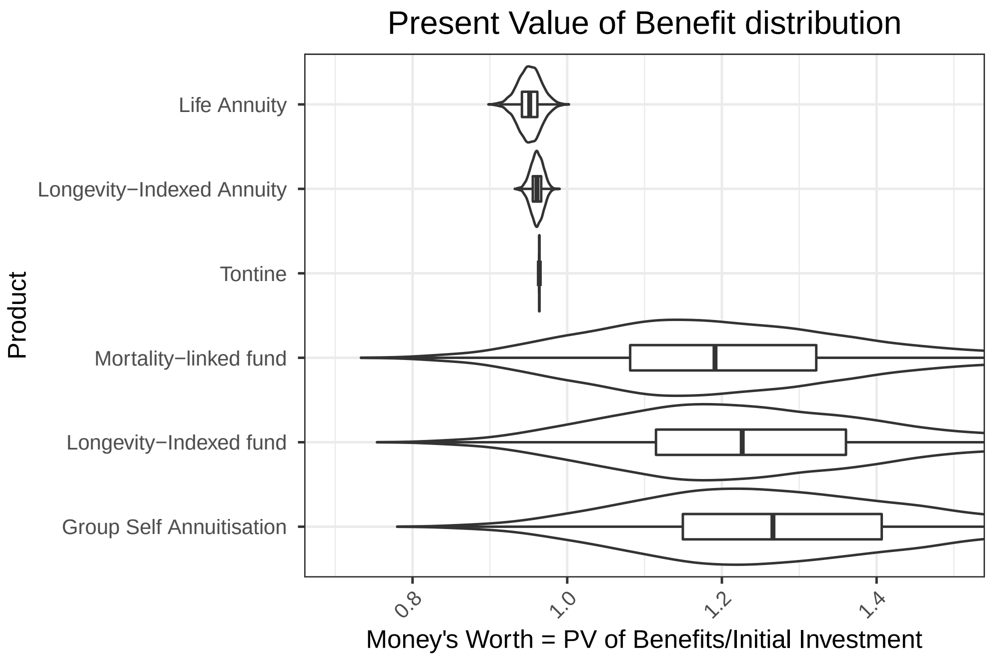

Figure 4.9: Distribution of actuarial present value for retirement income products

Table 4.5: Simulated money's worth

Product	Money's worth	Ranking
Life Annuity	0.95	7
Longevity-Indexed Annuity	0.96	6
Tontine	0.96	5
Account-based pension	0.96	4
Mortality-linked fund	1.19	3
Longevity-Indexed fund	1.22	2
Group Self Annuitisation	1.26	1

particularly as payments in earlier years are significantly less volatile. The right tails for the MLF, LIF and GSA are not shown, as they extend beyond the range of the plot.

4.6 Evaluation metrics

4.6.1 Actuarial present value

Recall in Subsection 3.4.2.1 that the actuarial present value (APV) is calculated with reference to the actual evolution of mortality of the pool of individuals who purchase the product across all simulations, with the use of a risk-free discount factor. Recall also that we defined the APV in percentage terms (APV%) as the APV divided by the initial purchase price, with the median of the APV% being the money's worth. The results of the APV% calculation are shown in Figure 4.9, with the corresponding money's worth reported in Table 4.5.

If the payout function of the product coincides with the definition of the APV, the APV will be a constant. That is to say, if the pool as a whole always receives the total amount they contribute over their collective lifetimes, as with a tontine, the APV is constant. In our case, we calculate the APV% to be 0.96, which is less than 1 because of the loading required for the financial guarantee. The distribution of the APV% for the LLLA is approximately centred on the tontine's APV%, albeit with higher variability. It is distributed between 0.93 and 0.99. This is because the policyholders receive payments according to the mortality evolution of a reference population, which is assumed to be large and diversified, but the pool itself is always finite. This small mismatch leads to the small variability in the APV%. The APV% for the life annuity has a wider distribution and is shifted to the left. It is distributed between 0.84 and 0.97, and has the lowest money's worth. This reflects two features of the contract. First, the wider distribution reflects the fact that the payments for the life annuity are set according to the best estimate of mortality at the beginning of the contract, and do not evolve according to the mortality experience. If aggregate mortality improves faster than expected, and the policyholders survive longer in aggregate, they receive a higher money's worth, and vice versa. Second, the more expensive financial guarantee reduces the money's worth since a charge is taken by the provider, shifting the distribution to the left.

We next consider the distributions for the three products with equity participation, the MLF, LIF and GSA. In Figure 4.9, the right tail is not shown because it is too large. **The GSA has the highest money's worth overall**. The MLF has the lowest money's worth of the three products. The effect of the mortality risk-sharing is analogous to the case of the products described above – the GSA has the widest distribution, while the MLF has the narrowest of the three.

For the purposes of this analysis, we model the ABP in the same way as other retirement income products. That is, we impose the assumption that the payments are contingent on survival and there is no bequest. This is clearly not how the ABP functions in reality, but this assumption is important in order for this analysis to be consistent with the results of the utility framework described in Subsection 4.6.3, as the utility framework we use also does not take into account any bequest payment. We also calculate the PV of all payments of the ABP, not conditioning on survival, which results in a far higher money's worth of 1.36, higher than the GSA.

4.6.2 AGA risk measure

The Australian Government Actuary (2018) risk measure (Equation (3.15)) is essentially a truncated semi-deviation with the deviation calculated with reference to the first payment of each product. This measure is expressed as a percentage of the first benefit, so its interpretation is similar to that of a coefficient of variation. Table 4.6 shows the ranking of the products.

Table 4.6: Australian Government Actuary risk measure

Product	Initial benefit ($)	AGA (%)	Rank
Life annuity	5872.46	0.00	1
LLLA	5932.39	5.99	5
Tontine	5950.92	10.31	7
MLF	5809.65	1.08	2
LIF	5981.14	1.34	3
GSA	6173.65	2.61	4
ABP	4000	9.93	6

The life annuity scores highest under this measure, since it never has any payments which fall below the initial payment. The two products which allow equity participation without pooling, the MLF, LIF and GSA, rank next. For the MLF and LIF, the equity participation decreases the chance that payments in the future will be lower than the initial payment, while the lack of pooling means that the payments are less likely to suddenly decline due to adverse mortality experience within the pool. For the GSA, despite the equity participation, the pooling mechanism means that payments at very advanced ages could be penalised under this measure. A finite pool has a large amount of idiosyncratic and systematic mortality risk at the most advanced ages (that is, at ages 105 and above), and this can lead to payments in the final years being greatly reduced compared to the initial benefit. The next product is the LLLA, as the passing of systematic mortality risk onto the policyholder means payments could fall below the initial value. The ABP also fares poorly, due to the high chance of fund exhaustion at old ages, which is due to the lack of a mortality premium. The worst product under the AGA risk measure is the tontine. The lack of equity participation leads to increased downside risk, while the pooling mechanism exacerbates the volatility of payments at very old ages.

This analysis highlights two limitations of the Australian Government Actuary risk measure. First, by considering only downside risk, it is very sensitive to the reference point, that is, the initial payment of each product. Therefore, contracts with guarantees, such as the life annuity, are likely to be favoured, even if the cost of meeting that guarantee is high. The MLF, LIF and GSA are ranked next. This reflects the fact that the MLF has more guarantees than the LIF and GSA, in spite of the fact that the MLF costs more than the LIF, which costs more than the GSA. This also leads to the counter-intuitive result that the higher the equity participation, the lower the risk measure, as the fund is likely to increase over time and there is less chance it falls below the initial payment. Second, the lack of survival probabilities means it is not holistic, and is unduly influenced by extreme events occurring at the oldest old ages. This means it is potentially unsuited to analyse pool-based products such as the GSA or tontine.

Table 4.7: Ranking and certainty equivalents of products under CRRA utility and habit formation

Product	CRRA CE	CRRA Rank	Habit CE	Habit Rank
Life annuity	5872.46	6	5872.46	6
LLLA	5921.34	5	5922.07	5
Tontine	5937.24	4	5937.64	4
MLF	7217.89	3	7394.39	3
LIF	7421.99	2	7607.12	2
GSA	7658.77	1	7854.48	1
ABP	5689.26	7	5790.44	7

4.6.3 Utility

In contrast to the previous metrics, a utility function trades off risk and return, taking into account survival probabilities. Thus, it allows for a more holistic evaluation of the product. The relevant formulae for the utility functions are given in Subsection 3.4.3.

Table 4.7 presents the evaluation of the products under constant relative risk aversion (CRRA) and utility with habit formation. We report the ranking of the products and their certainty equivalent.

First, for CRRA utility, we set $\rho = 2$ and $\beta = 0.98$, and ${}_tp_x$ as the mortality of a finite population of 1000 individuals. The relative risk aversion ρ and time preference parameter β are both set at a moderate level. Under this scenario, **the GSA scores highest**. This is due to the overwhelming gains that result from equity participation, along with the lack of a capital charge. The 'last survivor' effect may also play a role, however this is expected to be minor due to the low survival probability at extremely advanced ages. The next product is the LIF, followed by the MLF. These products score lower due to the loadings which are applied for the mortality risk. The products which rank lower still are the products which guarantee financial risk; the tontine, LLLA and annuity. Unsurprisingly, these are ranked in order of the cost of meeting the respective guarantees. The product to be ranked last is the ABP. This is because the CRRA utility function does not take into account the return of capital in such a product, and so understates the value of any product which has a bequest.

Recall that in our case, the certainty equivalent is calculated such that the individual would be indifferent between purchasing such a product and receiving a life annuity, where the level payment of the life annuity is equal to the certainty equivalent. This is an intuitive way to gauge the differences in the preferences of the products. Here, the large difference between products which allow equity participation and those which offer a guaranteed rate can be seen in monetary terms. This is again because of the high expected return assumptions we have set in the MLF, LIF and GSA, compared to the risk-free rate. For example, the difference between a GSA and a tontine is approximately $1402.58 of consumption per year under the

Table 4.8: Listing of sensitivity testing of various model elements

Model element	Parameter	Base case	Sensitivity
Pool size	L_x	1000	100
Policyholder's percentage of equity investment	α in R_t	30%	10%, 50%
Insurer's percentage of equity investment	α in R_t^I	10%	5%, 20%
Cost of capital	CoC	11%	0%, 7%, 15%
Mortality environment	q_x	q_x	$0.85 \times q_x$
Financial environment	μ	$\mu \approx 10.83\%$	$\mu + 0.02$, $\mu - 0.02$
Utility specification (CRRA)	$\rho,\ \beta$	2, 0.98	(5, 0.98), (2, 0.95)
Utility specification (Habit formation)	$\gamma,\ \lambda$	0.5, 0.5	(1, 1)

CRRA utility function.

For habit formation, we set $\gamma = 0.5$ and $\lambda = 0.5$, indicating that habit formation and consumption smoothing, respectively, are relatively important. Recall that we assume initial habit X_0 is the expected first benefit of a life annuity, similar to the Australian Government Actuary (2018) risk measure, where it is set to the expected first benefit of the product. The ranking of the products do not change and the certainty equivalent also does not change materially, indicating that that the results are robust to the specification of utility function.

4.7 Sensitivity

We now turn to the robustness of the model as a whole. Table 4.8 outlines the changes which will be made in the following subsections.

The changing of the mortality environment deserves further explanation. We assume that the simulated death probabilities q_x for the general population, and hence the finite pool of policyholders, are reduced by 15%. This change is similar to the longevity stress test prescribed by the Australian Prudential Regulation Authority, where the best estimate mortality rates are permanently reduced by 20% (APRA, 2012).

We assume that this is an *unexpected* future improvement in survivorship. We also assume that in products where the mortality risk is guaranteed, the provider does not adjust the contractual payments to policyholders. For the longevity-indexed products, we do assume the reference index changes according to this deviation, and therefore, the mortality credits are reduced. This implies the benefit paid to each policyholder will decline over time. For the products with risk pooling, similarly, payments to each policyholder will also decline over time.

In the insurer's fund, the formulation for the total value of actual survival benefits, $b_t^A = L_{x+t} b_t$ will be increased due to greater survivorship. This means that there will be an additional loading to cover the extra payments to the survivors, where these payments are guaranteed in

Table 4.9: Distribution of capital for an initial pool size of 100

Product	5%	25%	Median	75%	95%	97.5%	99%
MLF	-0.0482	-0.0185	0.0011	0.0202	0.0478	0.0563	0.0667
LLLA	-0.0430	-0.0171	0.0007	0.0181	0.0426	0.0509	0.0586
Life Annuity	-0.1504	-0.1046	-0.0722	-0.0400	0.0090	0.0236	0.0426
Tontine	-0.1495	-0.1037	-0.0728	-0.0422	0.0075	0.0216	0.0401
LIF	-0.1394	-0.0997	-0.0734	-0.0459	-0.0046	0.0098	0.0269
ABP	-0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
GSA	-0.0009	-0.0003	-0.0001	0.0000	0.0000	0.0000	0.0000

Table 4.10: Price as proportion of initial investment for an initial pool size of 100

Product	Price (%)
Life annuity	5.87
LLLA	5.52
Tontine	3.71
MLF	11.70
LIF	10.30
GSA	0
ABP	0

the contract.

We also change the financial environment to gauge the impact of a misestimation in the equity risk premium. The new expected return μ in the GBM is used to resimulate 5000 returns both for the policyholder's fund and insurer's fund.

4.7.1 The effect of pool size

This section will examine the impact of changing the initial pool size on the riskiness of the product and the perceived value to policyholders. The initial pool size is reduced to 100, and the resultant capital requirements, prices and ranking of products is computed.

Table 4.9 shows the effect of such a reduction. As the idiosyncratic mortality risk has been greatly increased, the products which guarantee this risk, namely the MLF, LLLA and life annuity have higher capital requirements. The MLF has the highest capital requirement since it guarantees both idiosyncratic and systematic mortality risk. However, as can be seen from the capital distribution of the LLLA, the majority of the capital is now held for idiosyncratic risk. Surprisingly, the annuity has a lower capital requirement. This is likely due to diversification benefits between the financial and longevity risk. The tontine has a moderately low capital requirement. The requirements for the ABP and GSA remain close to zero, as expected.

This riskiness is reflected in a far higher price for the MLF and LLLA, as shown in Table 4.10.

Table 4.11: Rankings of products for an initial pool size of 100

Product	Money's worth	AGA	Utility (CRRA)	Utility (CE)
Life annuity	7	1	6	5831.3
LLLA	6	5	5	5839.8
Tontine	5	7	4	5906.2
MLF	3	2	3	6904.5
LIF	2	3	2	6984.0
GSA	1	4	1	7671.5
ABP	4	6	7	5719.0

Table 4.12: Distribution of required capital where the insurer has 5% equity share

Product	5%	25%	Median	75%	95%	97.5%	99%
MLF	-0.0262	-0.0103	0.0003	0.0107	0.0262	0.0309	0.0359
Life annuity	-0.0807	-0.0550	-0.0370	-0.0196	0.0060	0.0147	0.0256
LLLA	-0.0142	-0.0057	0.0000	0.0057	0.0131	0.0157	0.0184
Tontine	-0.0764	-0.0533	-0.0373	-0.0219	0.0018	0.0092	0.0172
LIF	-0.0745	-0.0523	-0.0371	-0.0227	-0.0008	0.0052	0.0130
ABP	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
GSA	-0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 4.11 shows the evaluation of these products, incorporating the loadings. The rankings for all metrics have not changed compared to the base case.

4.7.2 The effect of insurer's investment strategy

Next we analyse the effect of changing the insurer's investment strategy on the riskiness and value of the products. We focus on comparing products where financial risk is guaranteed, namely, the life annuity, LLLA and tontine.

First, we present the effect of decreasing the insurer's investment in risky assets from 10% to 5%. This could reflect a scenario where the return on insurer's assets almost perfectly matches the financial returns which are promised to the policyholders, albeit with some frictional costs.

Table 4.12 shows the percentiles of the capital distribution under this scenario. Due to the improved financial strategy, the life annuity, LLLA and tontine have slightly lower capital requirements.

Table 4.13 shows the evaluation of these products, incorporating the loadings. The rankings for all metrics have not changed, though by comparing the CE, the life annuity, LLLA and tontine become more attractive compared to the base case.

Second, we present the case where the insurer invests in a higher proportion of equity, from 10% to 20%. The percentiles of the capital distribution can be found in Table 4.14. We can

Table 4.13: Rankings of products where the insurer has 5% equity share

Product	Money's worth	AGA	Utility (CRRA)	Utility (CE)
Life annuity	7	1	6	5963.5
LLLA	6	5	5	6019.7
Tontine	5	7	4	6050.6
MLF	3	2	3	7217.9
LIF	2	3	2	7422.0
GSA	1	4	1	7658.8
ABP	4	6	7	5689.3

Table 4.14: Distribution of required capital where the insurer has 20% equity share

Product	5%	25%	Median	75%	95%	97.5%	99%
Life annuity	-0.2482	-0.1803	-0.1299	-0.0774	0.0046	0.0316	0.0665
Tontine	-0.2480	-0.1795	-0.1299	-0.0791	0.0038	0.0284	0.0606
LLLA	-0.2477	-0.1797	-0.1302	-0.0785	0.0040	0.0300	0.0596
MLF	-0.0262	-0.0103	0.0003	0.0107	0.0262	0.0309	0.0359
LIF	-0.0142	-0.0057	0.0000	0.0057	0.0131	0.0157	0.0184
ABP	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
GSA	-0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

see the opposite effect compared to Table 4.12: the life annuity, LLLA and tontine have higher capital requirements due to the increased riskiness of providing the financial guarantee. This highlights the fact that the insurer must implement a sound matching strategy so that the riskiness of providing the guarantee is reduced as much as possible.

Furthermore, at the 99th percentile, the tontine requires more capital than the LLLA. This is because, in the case of the LLLA, the insurer can exploit the diversification benefit between financial and mortality risk to reduce the overall capital requirement. The increased capital requirement for the tontine is also reflected in a change to the preferred product in Table 4.15. Under the money's worth and CRRA utility measure, the tontine is now less preferred than the LLLA.

Table 4.15: Rankings of products where the insurer has 20% equity share

Product	Money's worth	AGA	Utility (CRRA)	Utility (CE)
Life annuity	7	1	7	5655.6
LLLA	5	5	4	5694.1
Tontine	6	7	6	5684.5
MLF	3	2	3	7217.9
LIF	2	3	2	7422.0
GSA	1	4	1	7658.8
ABP	4	6	5	5689.3

Table 4.16: Rankings of products where the policyholder has 10% equity share

Product	Money's worth	AGA	Utility (CRRA rank)	Utility (CRRA CE)
Life annuity	7	1	6	5872.5
LLLA	6	4	5	5921.3
Tontine	5	6	4	5937.2
MLF	3	2	3	6374.1
LIF	2	3	2	6524.6
GSA	1	5	1	6700.6
ABP	4	7	7	5009.6

4.7.3 The effect of policyholder's equity participation

In this section, we primarily consider contracts with equity participation (MLF, LIF and GSA), and consider their riskiness and value to the policyholder, as the proportion of equity investments held in the policyholder's fund changes from 30% to 10% and 50%.

Recall that the amount of capital required depends on both the total value of actual benefits paid to the policyholder and the financial strategy of the insurer (Equation (3.10)). For products with equity participation, we set the insurer's financial strategy to coincide with the policyholder's election. We also know that as the policyholder's equity participation increases, the insurer is liable to pay higher mortality credits as the fund value is higher. However, we find that the capital requirements are robust to changes in the policyholder's choice of equity investment proportion. For the MLF, the differences in the 99th percentile of capital are in the order of 10×10^{-10}. We do see a more substantial change in the price, as the price is calculated as a multiple of the expected fund value $E[F_t]$ at each time.

The evaluation of the products does not change markedly from the base case, as shown in Tables 4.16 and 4.17. However, we see that when the policyholder elects 50% equity participation, the ABP is ranked higher than all products with a financial guarantee, despite not taking into account the bequest. This shows that the financial guarantee is not valuable. The rankings also highlight the fact that even a small amount of equity participation (in the order of 10%) can make such a product more desirable compared to a product with a financial guarantee, due to the potential upside. We also report the certainty equivalents (CE) for the CRRA utility. By comparing the CE across differing proportions of equity participation, it can be seen that the products with higher equity participation are more preferred by individuals.

This analysis shows that product designers should be able to freely vary the proportion of equity participation in participating contracts, as the capital held by the provider remains unchanged. Any increase in the price due to the higher fund value is able to be passed on to policyholders, who would nevertheless still elect to purchase a product with higher proportion in equities.

Table 4.17: Rankings of products where the policyholder has 50% equity share

Product	Money's worth	AGA	Utility (CRRA rank)	Utility (CRRA CE)
Life annuity	7	1	7	5872.5
LLLA	6	6	6	5921.3
Tontine	5	7	5	5937.2
MLF	3	2	3	7854.6
LIF	2	3	2	8122.5
GSA	1	4	1	8433.8
ABP	4	5	4	6233.0

4.7.4 The effect of the cost of capital

We now turn to the effect of loadings on the desirability of the products by considering the effect of changing the cost of capital (CoC) from 11% to 0%, 7% and 15%. A higher cost of capital implies a higher price for the product. We analyse the unloaded case (CoC = 0%) to highlight the difference of imposing a charge for the riskiness of the guarantees. This is highlighted in the money's worth for the life annuity, as it decreases from 1 in the case of no loading, to 0.93 in the case of a 15% CoC.

Table 4.18 presents the evaluation of products under CRRA utility. Broadly speaking, the group of participating products rank higher than the group of products where the financial risk is guaranteed. The cost of the guarantee influences the ranking of the products within each group. We can see this by comparing the rankings of the life annuity with the LLLA and tontine; and the rankings of the MLF, LIF and GSA. Intuitively, if the policyholder is not charged for the risk, it is better to pass the risk on to the provider.

We can also see that at a point between CoC= 0% and CoC= 7%, the cost of the guarantee becomes too great and the order is reversed. A similar switching of the rankings due to the excessive cost of the guarantee was also shown in Table 4.15. This highlights the sensitivity of the policyholder's preferences to the cost of a guarantee in a retirement income product.

It is noteworthy that the life annuity is never the most preferred product as a whole, even in the case of no loadings for guarantees, in other words, actuarial fairness. This does not conflict with the literature on optimality of full annuitisation (see Yaari, 1965, Davidoff et al., 2005, for example), as we compare a life annuity to products which offer a variable, not fixed, rate of return.

4.7.5 The effect of misestimation of mortality

Now, we consider changes to the mortality environment, where the death probabilities for the reference population and the pool of policyholders are permanently reduced by 15% and thus, individuals who purchase the products survive longer than anticipated. The benefits

Table 4.18: Ranking of products under CRRA utility with differing cost of capital

Product	0%	7%	15%
Life annuity	4	6	6
LLLA	5	5	5
Tontine	6	4	4
MLF	1	3	3
LIF	2	2	2
GSA	3	1	1
ABP	7	7	7

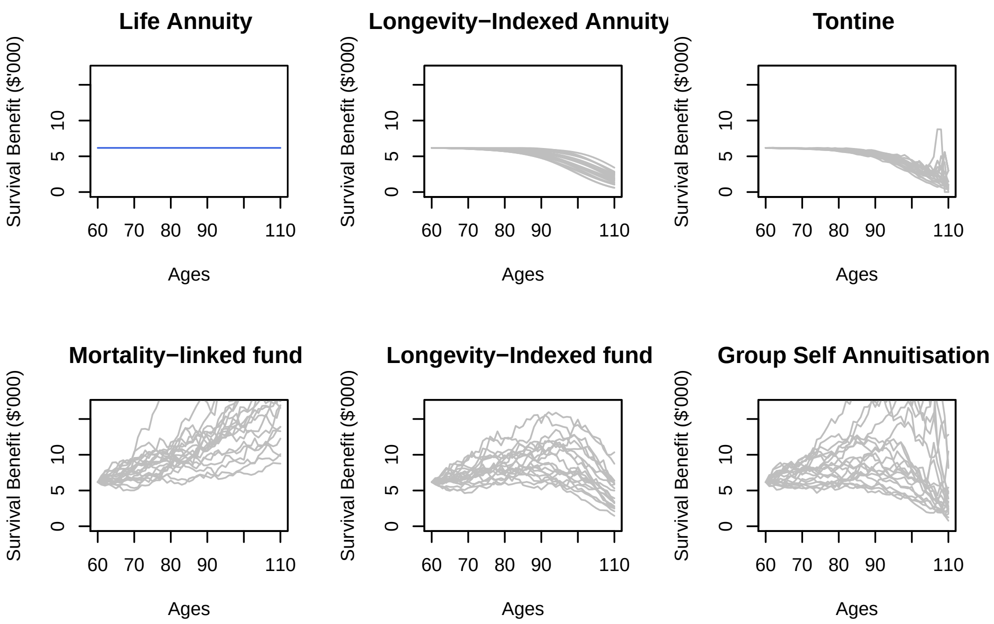

Figure 4.10: 20 simulations of survival benefits for retirement income products with unanticipated mortality improvements

promised, before loadings, for the life annuity and MLF are the same as in the base case. For the LLLA and LIF, the payments are assumed to evolve according to the changed mortality of the reference population, and hence will decline on average. The benefits for the pooled products, the tontine and GSA, evolve according to the pool, and hence these benefits will also decline on average. This is reflected in Figure 4.10.

The riskiness of these products is shown in Table 4.19. We see that products which have a full mortality guarantee are most risky. The life annuity ranks lower than the MLF at the 99th percentile because the insurer can exploit the diversification benefit between mortality and financial risk. This is also the case for the tontine and LLLA. We can also see that the capital requirements for longevity-indexed products (LLLA, LIF) are greatly reduced compared to the products with full mortality guarantees (annuity, MLF), as the provider passes the systematic mortality risk to the individual. Hence this is a reason why the provider would want to sell a

Table 4.19: Distribution of required capital with unanticipated mortality improvements

Product	5%	25%	Median	75%	95%	97.5%	99%
MLF	0.0056	0.0200	0.0302	0.0403	0.0544	0.0593	0.0635
Life annuity	-0.1165	-0.0772	-0.0467	-0.0150	0.0298	0.0442	0.0585
LLLA	-0.1386	-0.1008	-0.0719	-0.0423	-0.0008	0.0114	0.0283
Tontine	-0.1376	-0.1006	-0.0717	-0.0423	-0.0015	0.0108	0.0258
LIF	-0.0127	-0.0051	0.0000	0.0049	0.0121	0.0143	0.0164
GSA	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
ABP	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 4.20: Rankings of products with unanticipated mortality improvements

Product	Money's worth	AGA	Utility (CRRA)	Utility (CE)
Life annuity	7	1	5	5713.8
LLLA	6	7	7	5679.3
Tontine	5	6	6	5693.5
MLF	3	2	3	6972.1
LIF	2	4	2	7278.8
GSA	1	5	1	7476.3
ABP	4	3	4	5740.1

longevity risk-sharing product, as it requires less capital.

Next we consider the impact of this misestimation from the policyholder's perspective. First, according to the Australian Government Actuary (2018) risk measure, we can see that the LLLA ranks last, performing worse than the tontine and life annuity. This shows that retirees who are loss averse would not prefer this product. The CRRA utility measure shows that the life annuity ranks higher than the tontine and LLLA. We also note that the positions of the tontine and LLLA are swapped compared to the money's worth. This is counter-intuitive, but may be explained by the fact that the money's worth and CRRA use different assumptions regarding the discounting of cash flows. The money's worth uses the risk-free rate, while the CRRA utility uses a subjective time preference parameter.

This analysis shows that a product preferred by the provider may not be preferred by the individual. We find that the individual would rather pay to transfer the systematic mortality risk back to the provider (as in the case of a life annuity) rather than bear the risk themselves.

4.7.6 The effect of misestimation of financial returns

This section will now examine the impact of differing equity risk premia on the capital and value of the products.

First the equity risk premium (ERP) is reduced from 6.83% to 4.83%, which is likely a more realistic figure. The impact on the capital required is shown in Table 4.21. The three products

Table 4.21: Distribution of required capital for a reduction in the equity risk premium

Product	5%	25%	Median	75%	95%	97.5%	99%
Life annuity	-0.1221	-0.0798	-0.0497	-0.0196	0.0251	0.0393	0.0604
LLLA	-0.1197	-0.0791	-0.0497	-0.0206	0.0232	0.0371	0.0521
Tontine	-0.1196	-0.0783	-0.0496	-0.0216	0.0219	0.0340	0.0501
MLF	-0.0262	-0.0103	0.0003	0.0107	0.0262	0.0309	0.0359
LIF	-0.0142	-0.0057	0.0000	0.0057	0.0131	0.0157	0.0184
ABP	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
GSA	-0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 4.22: Ranking of products under CRRA utility with differing equity risk premium

Product	CE (low ERP)	Rank (low ERP)	CE (high ERP)	Rank (high ERP)
Life annuity	5700.0	6	6055.2	7
LLLA	5749.4	5	6104.6	5
Tontine	5761.5	4	6116.7	4
MLF	6750.0	3	7697.1	3
LIF	6925.7	2	7934.2	2
GSA	7130.1	1	8209.2	1
ABP	5315.3	7	6075.9	6

which guarantee financial risk have substantially higher capital charges than the products which provide no such guarantee. The whole capital distribution of these products has indeed been shifted to the right by approximately 2%, and hence on average the insurer makes approximately 2% less profit from selling each contract.

When the equity risk premium is increased from 6.83% to 8.83%, the capital distribution is analogously shifted to the left. For the life annuity, the capital required at the 99th percentile is reduced from 3.73% in the base case to 1.42% under this scenario.

In terms of desirability, we focus on evaluating the products using CRRA utility in Table 4.22. We find that under a low equity risk premium, the evaluation is the same as the base case. However, under a high equity risk premium, the equity participation becomes so valuable that the ABP is preferred over the life annuity, even though the CRRA utility does not take into account the bequest component.

4.7.7 The effect of utility specification

Finally, we perform more robustness checks on the specification of the utility functions.

First, we change the relative risk aversion parameter ρ from 2 to 5. This corresponds to varying levels of moderate risk aversion, with $\rho = 5$ representing a higher risk aversion than $\rho = 2$. The case of $\rho = 2$ is used in Hanewald et al. (2013) while the case of $\rho = 5$ is used in Stamos (2008) and Donnelly et al. (2013). This does not change the order of the ranking of the products, but

Table 4.23: Ranking of products under habit formation

Product	CE ($\gamma = 0$)	Rank ($\gamma = 0$)	CE ($\gamma = 1$)	Rank ($\gamma = 1$)
Life annuity	5872.5	6	5872.5	6
LLLA	5921.3	5	5927.1	5
Tontine	5937.2	4	5944.6	4
MLF	7217.9	3	7037.6	3
LIF	7422.0	2	7245.3	2
GSA	7658.8	1	7483.1	1
ABP	5689.3	7	5409.7	7

does lead to a tie between the MLF and LIF and between the annuity, LLLA and MLF.

The time preference β is changed separately from 0.98 to 0.95, representing an increased level of impatience, but this makes no difference to the ranking.

We also change the parameters in the habit formation utility function to more "extreme" levels. Recall that the parameter γ determines the importance of habit formation, with $\gamma = 1$ being very important and $\gamma = 0$ reducing to the case of CRRA utility. Also recall that λ determines the level of consumption smoothing, where $\lambda = 0$ corresponds to the case where only the previous period of consumption is considered, and $\lambda = 1$ corresponds to the case where only the initial reference habit is considered. We choose the parameter set $\gamma = 1,\ \lambda = 1$ to mimic the Australian Government Actuary (2018) risk measure, where here the habit is very important and the comparison is made to the initial habit X_0, the first expected payment of a life annuity.

It is perhaps most instructive to compare these results to the case of the CRRA utility (here we define it as the limiting case of habit formation, $\gamma = 0$). This is shown in Table 4.23. First, the specification of habit formation does not change the ranking of products. This has already been shown (with different parameters) in Subsection 4.6.3. Second, the differences in CE in the participating products (which have the highest volatility) between the two functions are minor overall, but not unsubstantial. It is counter-intuitive that the CE under the habit formation is higher than without habit formation. This could be due to the conservative choice of initial habit, which we assume to be the first payment of a (loaded) life annuity. For participating products, there is a high chance of exceeding the initial habit over the course of the retiree's lifetime and this may positively skew the habit formation metric.

CHAPTER 5

CONCLUSION

This chapter will present the key contributions of this thesis for three key groups: policymakers, the superannuation and life insurance industry, and academia. For policymakers and industry, we emphasise the results and implications. For academia, we focus on the contribution to the retirement incomes and modelling literature. We then present a number of suggestions for future work. This thesis concludes with a summary of the main contributions.

5.1 Implications for policymakers

5.1.1 The importance of risk pooling

The evaluation of the simulated benefits under utility metrics show the importance of incorporating risk pooling in a retirement income product. Considering the benefits conditional on survival, the account-based pension (ABP) ranks last under most economic scenarios, as it delivers a lower income throughout the retiree's lifetime.

This work also has implications for the evaluation of products which pool mortality risk, such as the group self annuitisation (GSA) and tontine. Recall that in such products, the provider does not guarantee longevity risk in any way. The account balance of these products do not exhaust until extremely advanced ages, meaning that risk pooling is likely to be a sustainable option, particularly when the prices for products with mortality guarantees are high.

Hence, this work affirms the fact that the presence of a product which incorporates risk pooling should be a component of the Comprehensive Income Product for Retirement (CIPR) (The Treasury, 2016). Furthermore, the advocacy of a pure pooled product without mortality guarantees, such as a GSA, or a product with mortality guarantees, such as a life annuity should depend on market competitiveness.

5.1.2 Australian Government Actuary risk measure

One recent example of a method to evaluate retirement income products can be found in the Australian Government Actuary (AGA) 2018 risk measure. This risk measure focuses on the downside risk of a product, relative to the initial benefit. Although it takes into account common behavioural heuristics such as loss aversion, it has a number of shortcomings.

Firstly, it is biased towards products with guarantees, without taking into the cost of their provision. This can be seen in the life annuity's first place ranking, as this is the product with the most guarantees, despite having the largest capital charge. The mortality-linked fund (MLF), longevity-indexed fund (LIF) and GSA are ranked 2nd, 3rd and 4th respectively. This reflects the fact that the MLF has more guarantees than the LIF and GSA, despite the MLF costing more than the LIF, which in turn costs more than the GSA. The cost of providing these guarantees is ignored because the deviation is calculated with respect to the initial payment, which by definition, is the initial payment *after* loadings have been added to reduce that payment. In an ideal world, the deviation could be calculated with reference to the actuarially fair initial benefit of a product. However, in practice, this is unlikely to be implemented, as the loadings are normally commercial-in-confidence.

Secondly, the AGA risk measure favours products with equity participation. This is because the higher expected returns of the equity markets lead to a reduced risk of a future payment falling below the initial payment[1]. This feature is not unique to the AGA risk measure, as a utility-based metric will also favour products which return a higher amount to the retiree on average through investment in the equity market. However, care must be taken when communicating this to retirees, as investment risk is understated.

Thirdly, it is unsuitable for the evaluation of pool-based products, particularly if they do not have a provision for the 'last survivor'. As the risk measure does not take into account survival probabilities, it is unduly influenced by extremely volatile payments due to uncertainty of the time of death of the last few survivors. In practice, however, this could be mitigated by a modified contract design, such as closing the pool once the number of survivors reach a minimum threshold.

[1] This has also been stated in Australian Government Actuary (2018)

5.1.3 The development of a modelling framework

One of the theoretical contributions of this thesis is the development of a consistent mathematical framework for modelling the guarantee structure in retirement income products (Appendix B.1). In Appendix B.2, we have also shown how to extend this framework to incorporate common features such as deferment, increasing payments and capital guarantees.

One of the key advantages of this framework is that it separates the various elements of the guarantee structure from the payout structure. This allows the development of new and hybrid products such as CIPRs by varying the elements of the fund equation. These hybrid products often have complex features such as deferment, and thus, Appendix B.2 is expected to aid in the evaluation of these products in the future.

To provide one concrete example, an example of a CIPR is the 'wrap', a combination of a deferred life annuity and an account-based pension (ABP). Using the relevant equations in Appendix B.2, we are able to write the guarantee structure of this product in a concise mathematical manner and easily compare it to both a life annuity and an ABP. We are then able to use the evaluation framework to determine the value of these products for the retiree.

5.2 Implications for industry

5.2.1 The product offering

Our results show that, where there is a lack of a guarantee, the financial risk faced by the policyholder is high throughout their lifetime, while the longevity risk is most acute at ages 80 and above. One solution is for the policyholder to purchase a product with guarantees.

However, we find that the associated cost of meeting the guarantee, which is passed on to the policyholder, is likely to be high enough in normal circumstances to dissuade retirees from purchasing a product with guarantees. Our simulations have shown that even with a modest cost of capital of 7%, the retiree is likely to prefer products with fewer guarantees. Retirees are likely to prefer to share the longevity risk in a pool (as in the case of the GSA) rather than pay to pass it fully or partially to a provider. However, when longevity risk is misestimated, such as when there is an unexpected improvement in longevity, the policyholder is more likely to pay for the longevity guarantee.

We also find that the ability of the policyholder to participate in equity markets is an overwhelmingly desirable feature in any retirement product. Our results show that even a small amount of participation, in the order of 10%, still makes a difference to the policyholder's evaluation of such products. The policyholder would still prefer to participate even when the proportion of equity investment is increased to 50%.

5.2.2 The design of pooled products

This work also has implications for the design of pooled products. As noted above, the 'last survivor' effect needs to be managed when the pool size reaches a certain minimum threshold. Furthermore, we find that although a small pool size (of 100) does not change the policyholder's preferences of the products, it does have a significant impact on the capital required. The amount of capital required dramatically increases in a product with longevity guarantees or risk-sharing, as the idiosyncratic risk of any one policyholder dying is magnified.

5.2.3 The loadings charged to policyholders

For a life insurer, the amount of capital required is crucial to the viability of a product, as the capital needs to be used efficiently to deliver a return for the company.

Our results show that the amount of capital required is sensitive to several factors:

- The financial and mortality guarantee structure;
- The pool size;
- The insurer's investment strategy; and
- The misestimation of the financial and mortality risk.

However, it is **not** sensitive to the policyholder's investment strategy in participating products.

This implies, firstly, that the provider of retirement income products must carefully consider the guarantee structure of the product and must devise an investment strategy suitable for that product in order to achieve a satisfactory capital requirement. For instance, for a life annuity, tontine and LLLA, a moderate reduction in capital can be achieved by almost perfectly matching the cash flows of the benefit payments. We simulated this in our work by reducing the proportion invested in equity by the insurer by 5%. However, for products which do not guarantee financial risk, no such consideration is necessary. Furthermore, the effect of misestimation of financial and/or mortality risk should not be underestimated. Naturally, the effect of this will depend on the guarantee structure of the product. Providers should also adjust their capital requirements regularly as the contract progresses.

Secondly, providers of participating products are free to offer a wide range of investment strategies to the policyholders, as in theory they should not need to change the amount of capital charged. Intuitively, this is because the financial risk is passed through to the policyholder, so there is no loading required by the provider.

5.3 Implications for academia

5.3.1 The development of a modelling framework

One of the major contributions of this thesis is the development of a comprehensive modelling framework which expresses a wide variety of financial and mortality guarantees in a consistent mathematical representation. This representation, called the fund equation, extends the analysis in Pitacco et al. (2009) and is presented in full in Appendix B.1 and Appendix B.2.

This framework can be used in future work to evaluate some of the hybrid products recently proposed in the academic literature, including the tonuity (Chen et al., 2018), a hybrid of the life annuity and tontine. This can be easily achieved due to the modular nature of the code developed, where the definition of the retirement income product consists of a series of functions to define the mortality credits, financial returns and benefit structure. Thus, to define a tonuity, one is able to easily modify and combine existing functions which have been written for the life annuity and tontine.

5.3.2 The development of an evaluation framework

The evaluation framework presented in this thesis extends the literature in the comparison of retirement income products.

Firstly, the evaluation of products extends the analysis of Hanewald et al. (2013) and Milevsky and Salisbury (2015) to incorporate realistic charges for varying financial and mortality guarantees. This is a crucial innovation, as it allows the comparison of products on a level playing field.

Secondly, this work extends the analysis of Piggott et al. (2005), Milevsky and Salisbury (2015) and Qiao and Sherris (2013) in that it incorporates both financial and mortality risk in a variety of pooled products and quantifies the relative importance of the two. We find that financial risk is more important for the provider, especially if they do not develop an adequate investment strategy.

Thirdly, to our knowledge, this work is one of the first to incorporate habit formation in the evaluation of retirement income products in a utility framework. Using habit formation allows us to examine behavioural features such as anchoring to a reference point, while examining the trade-off between risk and return. Although, we find the rankings of the products are not sensitive to the presence of habit formation, the differences in certainty equivalent utility highlight the importance of consumption smoothing and persistence of anchoring to the habit as important features of any future evaluation of retirement income products.

5.4 Future research

This research can be extended in a number of ways. First, the modelling could be extended to incorporate multiple cohorts. This would mean that people can enter the pool in later years. Piggott et al. (2005) consider this for the GSA, but it could be extended to other products with longevity risk sharing arrangements. Second, the issue of mortality heterogeneity could be considered. By assuming people have differing longevity risk, this could potentially impact the distribution of mortality credits in a retirement income product. A somewhat related issue is adverse selection in retirement income products. It is well known that an annuity is subject to adverse selection, where annuitants live longer than the general population. This has the potential to increase capital requirements for the insurer. Recently, the Actuaries Institute has published a paper giving sample mortality rates under adverse selection (Actuaries Institute, 2018). We leave the incorporation of this research to future work.

Dynamic strategies are also not considered. The capital is calculated only at one point in time, the inception. In reality, the capital should be updated yearly as experience emerges in the contract. This is the approach used in Solvency II (EIOPA, 2014). Hedging strategies for the financial risk borne by the insurer are not explicitly modelled. Furthermore, dynamic withdrawals are not considered for the account-based pension. Under the minimum withdrawal requirements, the withdrawal rate, and hence income, could be too low in early years of retirement (Balnozan, 2018).

Apart from modelling extensions to the retirement income products explained earlier, there is also scope to incorporate innovative retirement income product design. One example of this is in the development of products with long-term-care benefits.

In terms of the evaluation, the analysis could be extended by using more sophisticated mortality and financial models, improving the accuracy of the simulations. An economic scenario generator, such as that by Hanewald et al. (2013), can be used to incorporate inflation risk. More advanced models could be used to estimate the equity risk premium. The utility function could be extended to include a bequest motive (Bell et al., 2017), allowing it to be integrated with the economics literature more closely. Although we incorporate one non time-additive separable utility function through habit formation, we could extend this to more flexible cases such as Hyperbolic Absolute Risk Aversion or Epstein-Zin preferences (Backus et al., 2005). Finally, the project could incorporate the effect of the Age Pension means tests. The presence of the Age Pension has been shown to reduce the demand for life annuities (Iskhakov et al., 2015), but its effect is yet to be considered for other retirement income products. Incorporation of the Age Pension has the potential to make the work more applicable to the Australian context.

Finally, the demand side for such products could be explored further. When retirees purchase a retirement income product, they are essentially exchanging financial and longevity risk for default risk of the provider. This default risk, which is borne by the individual, has not been considered. Furthermore the role of subjective survival probabilities influencing the demand for retirement income products can also be incorporated in future work (see Chen et al., 2019, Weinert and Gründl, 2016).

5.5 Summary

In summary, the contribution of my thesis is both theoretical and practical.

1. I develop a modelling framework which achieves the following objectives:
    (a) comprehensively compares the guarantee structure of various retirement income products;
    (b) separates the guarantee structure from the payout structure, offering greater transparency in the communication of the product design;
    (c) simulates the fund equation and benefit payouts using appropriate stochastic mortality and financial models; and
    (d) applies appropriate loadings to products to reflect the cost of providing financial and longevity guarantees.
2. I develop an evaluation framework which extends previous evaluations of retirement income products. It:
    (a) quantifies the cost of providing financial and longevity guarantees;
    (b) use measures of risk and value to communicate desirability of products to industry and policymakers; and
    (c) extends the use of existing utility frameworks by incorporating habit formation into the evaluation decision.

APPENDIX A

FITTING OF GBM

First take a given series of monthly stock values $S_1 \dots S_n$. The log-return $x_i = \ln \frac{S_i}{S_{i-1}}$ each period follows a normal distribution:

$$x_i \sim \phi((\mu - \frac{1}{2}\sigma^2)T, \sigma T),$$

where μ and σ are the annual mean and volatility of the stock price, ϕ denotes the probability density function of the normal distribution, and $T = \frac{1}{12}$, since the stock returns are monthly.

Then we can solve the following equations simultaneously:

$$\begin{aligned} E[x_i] &= (\mu - \frac{1}{2}\sigma^2)T \\ \mathrm{Var}[x_i] &= \sigma^2 T, \end{aligned}$$

This yields the following estimates for μ and σ:

$$\begin{aligned} \hat{\mu} &= \frac{\hat{E}[x_i]}{T} + \frac{1}{2}\hat{\sigma}^2 \\ \hat{\sigma} &= \sqrt{\frac{\hat{\mathrm{Var}}[x_i]}{T}}, \end{aligned}$$

where $\hat{E}[x_i]$ represents the sample mean of the monthly stock log-return and $\hat{\mathrm{Var}}[x_i]$ represents the sample variance of the monthly stock log-return. This proof is adapted from Hull (2012).

APPENDIX B

DERIVATION OF THE FUND EQUATION FOR RETIREMENT INCOME PRODUCTS

B.1 Products used in this thesis

In this section, we derive the *mortality credits* Θ_t, the *financial returns* R_t and the *payout structure* b_t for all retirement income products that we consider in this work. It should also be noted that there are no loadings incorporated in these equations The notation used has been defined in Section 3.1.

B.1.1 Life annuity

We consider the fund equation for the whole portfolio of annuitants, presenting the work of Pitacco et al. (2009). At any time t, the total fund value is $l_{x+t}F_t$, where l_{x+t} is the best estimate of the number of individuals alive at time t. Note that in a life annuity, the R_t in Equation (3.9) is a constant r, since the annuitant receives a guaranteed financial return.

Therefore, using the notation in Section 3.1:

$$l_{x+t}F_t = l_{x+t-1}F_{t-1}(1+r) - l_{x+t}b_t, \qquad l_xF_0 = l_x(S-b_0), \tag{B.1}$$

$$F_t = \frac{l_{x+t-1}}{l_{x+t}}F_{t-1}(1+r) - b_t \qquad F_0 = S - b_0, \tag{B.2}$$

Now let $\frac{l_{x+t-1}}{l_{x+t}} = 1 + \Theta_t$.

$$\Theta_t = \frac{l_{x+t-1}}{l_{x+t}} - 1 = \frac{1}{p_{x+t-1}} - 1. \tag{B.3}$$

Θ_t is always greater than 0, so the mortality credit is always positive.

To find b_t we must use the prospective reserve :

$$\begin{aligned}
l_{x+t}F_{t^-} &= l_{x+t}(b_t + vp_{x+t}b_t + \cdots + v^{\omega-x-t}{}_{\omega-x-t}p_{x+t}b_t),\\
l_{x+t}F_{t^-} &= l_{x+t}\sum_{h=0}^{\omega-x-t} v^h \cdot b_t \cdot {}_hp_{x+t},\\
F_{t^-} &= b_t \cdot \ddot{a}_{x+t},\\
b_t &= \frac{F_{t^-}}{\ddot{a}_{x+t}},\\
b_t &= F_{t^-} \times \frac{1}{\ddot{a}_{x+t}} = F_{t^-} \times c(t;0) = b.
\end{aligned} \tag{B.4}$$

The first equation holds because the annuity is designed to give constant annual payments, so each benefit at time $t+1,\ t+2,\ \ldots\ ,\omega-x$, is the same as the benefit at time t, b_t. Furthermore, the probability of survival p_{x+t} and interest rate r are determined at inception of the contract, so the rule $c(t;0) = 1/\ddot{a}_{x+t}$ is similarly determined at inception. It can be shown that the benefit is constant for all t: $b_t = b$. It should be noted that if the best estimates of lives l_{x+t} and the risk-free rate $r_t = r$ is used, then the fund equation does not take into account any loadings.

In what follows, we adapt the derivations in Pitacco et al. (2009) to incorporate a wide variety of longevity and financial guarantees. It should also be noted that, similar to the assumption in Section 3.3, the equations presented in this section represent actuarially fair products; no loadings are incorporated into the analysis. The notation used here has been defined in Section 3.1.

B.1.2 Group self annuitisation

Following Piggott et al. (2005), the value of the fund for all participants for the case of the GSA can be expressed as:

$$\begin{aligned} L_{x+t}F_t &= L_{x+t-1}F_{t-1}(1+R_t) - L_{x+t}b_t, & L_xF_0 &= L_x(S-b_0) \\ F_t &= \frac{L_{x+t-1}}{L_{x+t}}F_{t-1}(1+R_t) - b_t & F_0 &= S - b_0 \end{aligned} \tag{B.5}$$

From the last line, it can be easily shown that $\Theta_t = \frac{L_{x+t-1}-L_{x+t}}{L_{x+t}}$.

These expressions hold because of the principle of mutuality: only the survivors share in the gains of the fund, whereas the deceased lose their share of the fund. It is analogous to the case of the life annuity, as shown in the previous section, with the deterministic l_x replaced with a stochastic L_x.

The payout structure at time t can be computed by considering the total fund value $L_{x+t}F_{t-}$ spread equally across each remaining survivor across their expected lifetime, which is determined at the beginning of the contract. This will allow us to calculate the fund value using a prospective reserving argument, similar to that for the life annuity. Piggott et al. (2005) show that this approach is mathematically equivalent to adjusting the benefit each period to account for deviations in experience, as in Equation (2.2). The proof is as follows:

$$\begin{aligned} L_{x+t}F_{t^-} &= L_{x+t}(b_t + vp_{x+t}b_t + \cdots + v^{\omega-x-t}{}_{\omega-x-t}p_{x+t}b_t) \\ L_{x+t}F_{t^-} &= L_{x+t}\sum_{h=0}^{\omega-x-t} v^h \cdot b_t \cdot {}_hp_{x+t} \\ F_{t^-} &= b_t \cdot \ddot{a}_{x+t} \\ b_t &= F_{t^-} \times \frac{1}{\ddot{a}_{x+t}} = F_{t^-} \times c(t;0). \end{aligned}$$

The benefits here are not constant across time as it depends on the fund value F_{t^-}, which in turn depends on the actual number of survivors at any point t: L_{x+t} as well as the financial return earned between time $t-1$ and t: R_{t-1}. Furthermore, since the probability of survival p_{x+t} at each time t is determined at inception, the rule $c(t;0) = 1/\ddot{a}_{x+t}$ is similarly determined at inception.

B.1.3 Tontine

In a tontine, the financial returns are guaranteed by the provider instead of being shared among the pool, hence R_t is replaced by r_t in Equation (B.5):

$$F_t = \frac{L_{x+t-1}}{L_{x+t}} F_{t-1}(1+r) - b_t \qquad\qquad F_0 = S - b_0$$

In Milevsky and Salisbury (2015) the payout for a natural tontine (in our notation) is defined as:

$$b_t = \frac{(Sl_x)({}_tp_x)}{L_{x+t}\ddot{a}_x} \tag{B.6}$$

This offers a different perspective on the design of a product with longevity risk pooling. A set of rules is set up initially – the quantities l_x, S, and $\ddot{a}_x$ are determined at inception and cannot be changed. Then the benefit evolves according to one term only: the (stochastic) number of survivors at time t, L_{x+t}. The disadvantage of this approach is that the fund value is no longer present in Equation (B.6) – risk management from the perspective of the provider is made more difficult. Fortunately, it can be shown that the formulation in Equation (B.6) is identical to the formulation of the GSA's fund equation and payout structure, in the case of a deterministic financial return. It can also be shown that the tontine payout can be expressed in terms of the recursive benefit formulation in Piggott et al. (2005). The proofs for these two results can be found below.

Theorem B.1.1 *The payout function* $b_t = \frac{Sl_x({}_tp_x)}{L_{x+t}\ddot{a}_x}$ *is equivalent to* $b_t = F_{t^-} \times c(t;0)$ *where* $F_{t^-} = \frac{L_{x+t-1}}{L_{x+t}}(1+r)F_{t-1}$ *and* $c(t;0) = \frac{1}{\ddot{a}_{x+t}}$.

Proof. Recall the fund equation for a GSA at time $t = 1$. We set a deterministic financial return r for consistency with the tontine:

$$\begin{aligned}
F_1 &= \frac{l_x}{L_{x+1}} F_0(1+r) - b_1 \qquad\qquad\qquad\qquad\qquad\qquad F_0 = S - b_0 \\
b_0 &= \frac{S}{\ddot{a}_x} \\
b_1 &= \frac{F_{1-}}{\ddot{a}_{x+1}} \\
&= \frac{l_x}{L_{x+1}} F_0(1+r) \frac{1}{\ddot{a}_{x+1}} \\
&= \frac{l_x}{L_{x+1}} \left(S - \frac{S}{\ddot{a}_x}\right)(1+r) \frac{1}{\ddot{a}_{x+1}} \\
&= \frac{Sl_x}{L_{x+1}}(1+r)\left(\frac{\ddot{a}_x - 1}{\ddot{a}_x}\right) \frac{l_{x+1}}{(1+r)(\ddot{a}_x - 1)l_x} \\
&= \frac{Sl_{x+1}}{L_{x+1}} \frac{1}{\ddot{a}_x} \\
&= \frac{(Sl_x)(p_x)}{L_{x+1}\ddot{a}_x}
\end{aligned}$$

In the third last line we use the well-known actuarial identity $1 + vp_x\ddot{a}_{x+1} = \ddot{a}_x$, solving for $\ddot{a}_{x+1}$. We have hence proven the Equation (B.6) for the case of $t = 1$. By induction it holds also for a general t. □

Theorem B.1.2 *The payout function* $b_t = \frac{Sl_x({}_tp_x)}{L_{x+t}\ddot{a}_x}$ *is equivalent to* $b_t = b_{t-1} \times MEA_t$ *where* $MEA_t = \frac{p_{x+t-1}}{P_{x+t-1}} = \frac{l_{x+t}}{l_{x+t-1}} \times \frac{L_{x+t-1}}{L_{x+t}}$.

Proof.

$$b_t = \frac{Sl_x({}_tp_x)}{L_{x+t}\ddot{a}_x} \tag{B.7}$$

$$b_{t-1} = \frac{Sl_x}{\ddot{a}_x}({}_{t-1}p_x)\left(\frac{1}{L_{x+t-1}}\right) \tag{B.8}$$

Solve for $\frac{Sl_x}{\ddot{a}_x}$ in Equation (B.8):

$$\begin{aligned}
\frac{Sl_x}{\ddot{a}_x} &= \frac{b_{t-1}}{{}_{t-1}p_x} L_{x+t-1} \longrightarrow \text{(B.7)} \\
b_t &= \frac{b_{t-1}({}_tp_x)}{{}_{t-1}p_x} \frac{L_{x+t-1}}{L_{x+t}} \\
&= b_{t-1} \frac{l_{x+t}}{l_x} \frac{l_x}{l_{x+t-1}} \frac{L_{x+t-1}}{L_{x+t}} \\
&= b_{t-1} \frac{p_{x+t-1}}{P_{x+t-1}}
\end{aligned}$$

Now for the converse:

$$\begin{aligned}
b_t &= b_{t-1} \times \frac{l_{x+t}}{l_{x+t-1}} \frac{L_{x+t-1}}{L_{x+t}} \\
&= b_{t-2} \times MEA_{t-1} \times MEA_t \\
&= b_{t-2} \times \frac{l_{x+t}}{L_{x+t}} \frac{L_{x+t-2}}{l_{x+t-2}} \\
&= \dots \\
&= b_1 \times \frac{l_{x+t}}{L_{x+t}} \frac{L_{x+1}}{l_{x+1}} \\
&= b_0 \times \frac{l_{x+t}}{L_{x+t}} \frac{l_x}{l_x} \\
&= \frac{Sl_x({}_tp_x)}{\ddot{a}_x L_{x+t}}
\end{aligned}$$

$\square$

B.1.4 Longevity-indexed life annuity

In a longevity-indexed life annuity (LLLA) (Denuit et al., 2011), the longevity risk is shared between the provider and individual, where the payments depend on the setting of a reference population l_{x+t}^{ref} at inception of the contract. As a result, if the actual mortality rate deviates from the reference population, this is borne by the policyholders. The fund equation is given by:

$$\begin{aligned}
l_{x+t}^{ref} F_t &= l_{x+t-1}^{ref} F_{t-1}(1+r) - l_{x+t}^{ref} b_t, & l_x^{ref} F_0 &= l_x^{ref}(S - b_0) \\
F_t &= \frac{l_{x+t-1}^{ref}}{l_{x+t}^{ref}} F_{t-1}(1+r) - b_t & F_0 &= S - b_0 \\
\Theta_t &= \frac{l_{x+t-1}^{ref} - l_{x+t}^{ref}}{l_{x+t}^{ref}} \\
b_t &= F_{t^-} \times \frac{1}{\ddot{a}_{x+t}^{ref}} = F_{t^-} \times c(t;0).
\end{aligned}$$

B.1.5 Mortality-linked fund

Next, we consider a product where the mortality credits paid to the policyholder are deterministic, as in the case of a life annuity, but the fund is invested in risky assets. This design is similar to the mortality-linked fund defined in Section 2 (Donnelly et al., 2013). For the purposes of this analysis, this product will simply be called the mortality-linked fund (MLF), and the equations defining this product are shown below:

$$\begin{aligned}
l_{x+t}F_t &= l_{x+t-1}F_{t-1}(1+R_t) - l_{x+t}b_t, & l_xF_0 &= l_x(S-b_0)\\
F_t &= \frac{l_{x+t-1}}{l_{x+t}}F_{t-1}(1+R_t) - b_t & F_0 &= S-b_0\\
\Theta_t &= \frac{l_{x+t-1}-l_{x+t}}{l_{x+t}}\\
b_t &= F_{t^-} \times \frac{1}{\ddot{a}_{x+t}} = F_{t^-} \times c(t;0).
\end{aligned}$$

B.1.6 Longevity-indexed fund

Recall that we have introduced a new product design which combines features of both the longevity-indexed life annuity and mortality-linked fund. In this design, the policyholders receive mortality credits according to a longevity-indexed life annuity, while the fund is invested in risky assets. This will be called the *longevity-indexed fund* (LIF), whose fund equation is given by:

$$\begin{aligned}
l^{ref}_{x+t}F_t &= l^{ref}_{x+t-1}F_{t-1}(1+R_t) - l^{ref}_{x+t}b_t, & l_xF_0 &= l_x(S-b_0)\\
F_t &= \frac{l^{ref}_{x+t-1}}{l^{ref}_{x+t}}F_{t-1}(1+R_t) - b_t & F_0 &= S-b_0\\
\Theta_t &= \frac{l^{ref}_{x+t-1}-l^{ref}_{x+t}}{l^{ref}_{x+t}}\\
b_t &= F_{t^-} \times \frac{1}{\ddot{a}^{ref}_{x+t}} = F_{t^-} \times c(t;0).
\end{aligned}$$

B.1.7 Phased withdrawal

Consider a phased withdrawal product structured such that a percentage of the fund value γ_t is withdrawn at the end of each year, so that $b_t = F_{t^-} \times c(t;0) = F_{t^-}\gamma_t$.

As there is no longevity risk sharing, the fund equation can be written from the perspective of one individual, simplifying it considerably:

$$\begin{aligned}
F_t &= F_{t-1}(1+R_t) - \gamma_t F_{t^-} & F_0 &= S-b_0\\
F_t &= F_{t-1}(1+R_t)(1-\gamma_t).
\end{aligned}$$

B.2 Additional product features

This section considers the derivation of various features which could be added to the retirement income products presented in the previous section. For simplicity, the features, namely, deferment, non-constant payments and capital guarantees are incorporated with a life annuity as the underlying product. It should also be noted that there are no loadings incorporated in these equations.

B.2.1 Deferment

Let the deferment period of the deferred life annuity be m years. That is, conditional on survival, there is no payment from $0 < t < m$, with the first payment from time m continuing annually to $\omega - x$.

Therefore the fund equation can be stated as:

$$\begin{aligned} l_{x+t}F_t &= l_{x+t-1}F_{t-1}(1+r) & l_xF_0 &= l_x(S), \quad 0 < t < m \\ F_t &= \frac{l_{x+t-1}}{l_{x+t}}F_{t-1}(1+r) & F_0 &= S, \quad 0 < t < m \\ F_t &= \frac{l_{x+t-1}}{l_{x+t}}F_{t-1}(1+r) - b_t \quad m & &\leq t < \omega. \end{aligned}$$

It can be thus be easily seen that Θ_t is the same as Equation (B.3) for all time t: $\Theta_t = \frac{l_{x+t-1} - l_{x+t}}{l_{x+t}}$.

To calculate b_t, firstly note that there is no payment during the deferment period $0 < t < m$, hence, there is no benefit. After the deferment period m, the payout structure is also the same as an ordinary life annuity since the prospective reserve is the same. Hence,

$$b_t = \begin{cases} 0, & 0 \leq t < m, \\ \frac{F_{t^-}}{\ddot{a}_{x+t}}, & m \leq t < \omega. \end{cases}$$

B.2.2 Non-constant payments

Consider the case where the life annuity is modified to give payments which vary across time according to an index i (for example, the Consumer Price Index). Therefore the benefit would be defined recursively: $b_t = b_{t-1}(1 + i_{t-1})$. We have the usual fund equation:

$$\begin{aligned} l_{x+t}F_t &= l_{x+t-1}F_{t-1}(1+r) - l_{x+t}b_t, & l_xF_0 &= l_x(S - b_0) \\ F_t &= \frac{l_{x+t-1}}{l_{x+t}}F_{t-1}(1+r) - b_t. & F_0 &= S - b_0 \end{aligned}$$

To find b_t we use the prospective reserve:

$$\begin{aligned}
l_{x+t}F_{t^-} &= l_{x+t}\left(b_t + vb_t(1+i_t)p_{x+t} + \cdots + v^{\omega-x-t}b_t\prod_{j=t}^{\omega-x-1}(1+i_j)_{\omega-x-t}p_{x+t}\right)\\
l_{x+t}F_{t^-} &= l_{x+t}\left(b_t + \sum_{h=1}^{\omega-x-t}\left(v^h\ b_t\prod_{j=t}^{h+t-1}(1+i_j)\ {}_hp_{x+t}\right)\right)\\
F_{t^-} &= b_t + b_t\sum_{h=1}^{\omega-x-t}\left(v^h\prod_{j=t}^{h+t-1}(1+i_j)\ {}_hp_{x+t}\right)\\
b_t &= \frac{F_{t^-}}{1+\sum_{h=1}^{\omega-x-t}\left(v^h\ \prod_{j=t}^{h-1}(1+i_j)\ {}_hp_{x+t}\right)}.
\end{aligned}$$

If we make the simplifying assumption that i_t is constant and known in advance we can simplify the denominator. Let $i_t = i^*$. Then:

$$\begin{aligned}
1+\sum_{h=1}^{\omega-x-t}\left(v^h\ b_t\prod_{j=t}^{h-1}(1+i_j)\ {}_hp_{x+t}\right) &= 1+\sum_{h=1}^{\omega-x-t}\left(v^h\ b_t(1+i^*)^h\ {}_hp_{x+t}\right)\\
&= 1+\sum_{h=1}^{\omega-x-t}\left(v^{h*}\cdot b_t\cdot {}_hp_{x+t}\right)\\
&= \ddot{a}^*_{x+t},
\end{aligned}$$

where $v^* = \frac{1+i^*}{1+r}$. Hence $b_t = \frac{F_{t^-}}{\ddot{a}^*_{x+t}}$.

B.2.3 Capital guarantees

There are multiple ways to define a life annuity which incorporates a capital guarantee, which returns some capital to the annuitant at the end of the year of death, if the annuitant dies within the first n years. When analysing a capital guarantee, the fund equation must be modified to take into account this death benefit. Consider firstly the case when the capital guarantee is in operation, from $0 \le t < n$:

$$\begin{aligned}
l_{x+t}F_t &= l_{x+t-1}F_{t-1}(1+r) - l_{x+t}b_t - (l_{x+t-1}-l_{x+t})I_t, & l_xF_0 &= l_x(S-b_0), & 1 \le t \le n,\\
F_t &= \frac{l_{x+t-1}}{l_{x+t}}F_{t-1}(1+r) - b_t - \frac{l_{x+t-1}-l_{x+t}}{l_{x+t}}I_t, & F_0 &= S-b_0, & 1 \le t \le n,
\end{aligned} \tag{B.9}$$

where b_t here is the survival benefit and I_t is the death benefit.

When the capital guarantee is not in operation, from time $t \ge n$, the fund equation reduces to that of an ordinary life annuity, in Equation (B.1).

In both cases, we can see Θ_t can be determined in the same way for all time t as Equation

(B.3), $\Theta_t = \frac{l_{x+t-1}-l_{x+t}}{l_{x+t}}$.

To solve for b_t for the periods $0 \leq t \leq n$ we can consider the prospective reserve:

$$\begin{aligned}
F_{t^-} &= b_t + vp_{x+t}b_t + v^2{}_2p_{x+t}b_t + \cdots + v^{\omega-x-t}{}_{\omega-x-t}p_{x+t}b_t + \\
&\quad vq_{x+t}I_{t+1} + v^2{}_{1|}q_{x+t}I_{t+2} + \ldots v^{n-t}{}_{n-t-1|}q_{x+t}I_n \\
F_{t^-} &= \sum_{h=0}^{\omega-x-t} v^h\ b_t\ {}_hp_{x+t} + \sum_{h=1}^{n-t} v^h I_{t+h}\ {}_{h-1|}q_{x+t} \\
F_{t^-} &= b_t\ddot{a}_{x+t} + \sum_{h=1}^{n-t} v^h I_{t+h}\ {}_{h-1|}q_{x+t} \\
F_{t^-} &= b\ \ddot{a}_{x+t} + \sum_{h=1}^{n-t} v^h I_{t+h}\ {}_{h-1|}q_{x+t}
\end{aligned}$$

where the first line of the first equation signifies the survival benefit term b_t and the second line signifies the death benefit terms $I_{t+1}, I_{t+2}, \cdots$ We need the assumption of constant benefit $b_t = b$ to specify the death benefit I_t in the last equation. In the last equation we can let the term inside the summation be $(GA)^{1}_{x+t:\overline{n}|}$ to denote a term insurance with variable benefit for n years.

In the case of $t > n$, the benefit is the same as that of a life annuity. Hence we can combine the previous two results to give an expression for the benefit $b_t = b$:

$$b = \begin{cases} \frac{F_{t^-} - (GA)^{1}_{x+t:\overline{n}|}}{\ddot{a}_{x+t}} & 0 \leq t \leq n, \\ \frac{F_{t^-}}{\ddot{a}_{x+t}}, & n < t < \omega. \end{cases}$$

We now turn to the payout structure of the death benefit, I_t. If the death benefit I_t can be written as a function of the fund reserve F_t, the benefit structure and fund equation (B.9) can be simplified considerably. However, in practice, the death benefit depends on the amount paid out already, or depends on the initial capital invested.

In one such setting by Boardman (2006), this product, called a money-back annuity is defined such that the initial capital S is returned upon death to the annuitant, minus the sum of nominal payments made so far in the contract. We set a time limit on the capital guarantee, in contrast to the original paper where the initial capital can be returned until the sum of the payments made so far exceeds the initial capital. The formula is given by:

$$I_t = \begin{cases} \max(S - bt, 0), & 1 \leq t \leq n, \\ 0, & t > n. \end{cases} \tag{B.10}$$

However this is difficult to implement in our setting since the max function would require numerical methods to calculate b.

We further assume that t is not too large, and that $S > bt$. We also know that $b = \frac{S}{\ddot{a}_x}$ by substituting $t = 0$ in Equation (B.4). Using this, we can rewrite (B.10) as:

$$I_t = \begin{cases} S - S\,\frac{1}{\ddot{a}_x} \times t, & 1 \leq t \leq n, \\ 0, & t > n. \end{cases} \tag{B.11}$$

An alternative specification is to write the death benefit **as a function of the initial capital and time**, rather than the benefit paid out. This has been adopted by Comminsure (2017) and Challenger (2019). This results in the following respective formulations for I_t:

$$I_t = \begin{cases} S - S\,\frac{1}{n} \times t, & 1 \leq t \leq n, \\ 0, & t > n, \end{cases} \tag{B.12}$$

$$I_t = \begin{cases} S - S\,\frac{1-\alpha}{15} \times t, & 1 \leq t \leq 15,\ 0 < \alpha < 1, \\ 0, & t > 15, \end{cases} \tag{B.13}$$

where the α is decided upon inception of the contract.

We can see the similarities in equations (B.10)-(B.13) and we propose a general formulation for I_t:

$$I_t = \begin{cases} S - S\beta \times t, & 1 \leq t \leq n, \\ 0, & t > n. \end{cases} \tag{B.14}$$

We can also examine a special case of the life annuity with capital guarantees where the fund equation simplifies considerably. This is where the death benefit is a **function of the past reserve**. For the first n years, there is no risk pooling between participants, and no survival benefits. That is, at the end of the year of death, the participant receives their investment thus far in the fund: $I_t = F_{t-1}(1 + r)$. Then the fund equation for the first n years (B.9) becomes:

$$
\begin{aligned}
l_{x+t}F_t &= l_{x+t-1}F_{t-1}(1+r) - I_t(l_{x+t-1} - l_{x+t}) && 1 \leq t \leq n \\
F_t &= \frac{l_{x+t-1}}{l_{x+t}}F_{t-1}(1+r) - \left(\frac{l_{x+t-1}}{l_{x+t}} - 1\right)(F_{t-1}(1+r)) \\
F_t &= \frac{l_{x+t-1}}{l_{x+t}}F_{t-1}(1+r) - \frac{l_{x+t-1}}{l_{x+t}}F_{t-1}(1+r) + F_{t-1}(1+r) \\
F_t &= F_{t-1}(1+r)
\end{aligned}
$$

This is exactly the same as a bond for the first n years. There are no mortality credits: $\Theta = 0$. After this period, the guarantee structure and payout is the same as a (deferred) life annuity.

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