Robust Estimation of Dynamic Models with Unmodelled Errors

An Instrumental Variable Approach

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Introduction

Objectives of Research

The aim is to examine and compare the consistency of estimators in dynamic regression models. Commonly, the estimators in such models are rendered inconsistent by the presence of serial correlation in the errors. There is no generally accepted response to take in such cases. Current methods employ an autocorrelation correction approach, such as the Cochrane-Orcutt procedure.

However, in light of the growing body of literature criticising the practice of autocorrelation correcting, and the inconsistencies and inefficiencies that have been shown to follow even under strong regularity conditions, the confidence instilled into practitioners by the availability of this method has been replaced by wariness.

This research takes a different approach, by examining whether the method of instrumental variable (IV) estimation can be employed to achieve consistent estimation. IV estimation is commonly employed outside of the time series context, but has been largely ignored within time series literature because the instruments are theoretically invalid. However, the research and simulations undertaken here show that this does not preclude them from providing consistent estimates. In particular, the degree of inconsistency incurred when using strictly invalid instruments is shown to be much lower than that incurred under existing methodology.

Significance and Motivation

Time series data comprises a large proportion of any financial or economic analysis. The standard dynamic model examined in this thesis is the building block for many more complex models used by practitioners. It is hoped that by providing a simple and prescriptive way to consistently estimate such models, the pitfalls inherent in dynamic modelling will be lessened, and more reliable results will follow. This research was motivated by the large body of literature that criticises current dynamic estimation methods, which has yet to have impacted significantly outside of academic circles.
Research Approach

Models and Parameterisation

Several types of dynamic models were simulated:

1. Simple Autoregressive Model:

   \[ y_t = \alpha y_{t-1} + u_t \]

2. Autoregressive Distributed Lag Model

   \[ y_t = \alpha y_{t-1} + \beta_1 x_t + \beta_2 x_{t-1} + u_t \]

In each of the above models, \( u_t \) represents the error term, \( x_t \) and \( x_{t-1} \) are exogenous explanatory variables and \( y_{t-1} \) is an endogenous explanatory variable. These two basic models are simplified representations of the most commonly employed dynamic regression models in practice.

The estimation of the parameters of interest \( \alpha, \beta_1 \) and \( \beta_2 \) is complicated by the error term. When the error term \( u_t \) contains some (unknown) type of serial correlation, the parameter estimates will be inconsistent when estimated by ordinary least squares or even by autocorrelation-correction methods. To see the extent of the inconsistency, several of the most common forms of serial correlation were also examined:

1. Autoregressive (AR) Errors:

   \[ u_t = \rho u_{t-1} + e_t \]

2. Moving Average (MA) Errors:

   \[ u_t = \rho e_{t-1} + e_t \]

3. Fractionally Integrated Errors:

   \[ u_t = \sum_{k=0}^{\infty} \frac{\Gamma(k + d)}{\Gamma(d)\Gamma(k + 1)} e_{t-k} \], where \( \Gamma(\cdot) \) represents the gamma function.

In each of the above, \( e_t \) represents white-noise disturbances, and \( \rho \) is the "nuisance" parameter of serial correlation persistence. There are several tests that can detect serial correlation in the residuals of regression models, but importantly, none of these tests are
able to distinguish the particular type. The purpose of this research is to design a method to estimate the parameters of interest \( (\alpha, \beta) \) which does not require knowledge of the type of serial correlation or the nuisance parameter \( \rho \). This will enable practitioners to estimate the models simply and consistently.

**Estimation and Simulation Method**

The inconsistency of ordinary least squares estimates essentially arises from the fact that the explanatory variables are correlated with the errors by way of the serial correlation structure. The method of instrumental variable (IV) estimation overcomes this by replacing the explanatory variables with suitable instruments. These must be correlated with the explanatory variables, but not with the errors. In the time series context, an obvious set of instruments comprises lags of the explanatory variables:

\[
Z_t = \{x_{t-4}, x_{t-5}, x_{t-6}, \ldots, y_{t-4}, y_{t-5}, y_{t-6}, \ldots\}
\]

The exact composition of this set of instruments was considered extensively through the course of research. The number of instruments to use is a logarithmic function of the sample size: so if there are \( T \) time periods of data points, then the optimal number of instruments was found to be \( \ln(T) \) lags of each of the explanatory variables. For a simple autoregressive distributed lag model, which contains one lagged \( y \) regressor and one lagged \( x \) regressor, the set of instruments consists of \( \ln(T) \) lags of each variable, beginning at \( x_{t-4} \) and \( y_{t-4} \). This is because the usefulness of instruments depends on both their strength and their validity, and a four-period lag was found to provide the best balance between these two.

The new model is estimated as using a version of Generalised Instrumental Variable (GIV) estimation. The performance of the GIV estimators can be optimised with respect to the set of instruments \( Z_t \) (including both its components and its dimension) and a weighting matrix that assigns more importance to the elements of \( Z_t \) that are weakly correlated with \( u_t \):

\[
\begin{align*}
\left(\alpha, \hat{\beta}_1, \hat{\beta}_2\right) &= \left(X'ZM^{-1}Z'X\right)^{-1}X'ZM^{-1}Z'y \\
\text{where} \quad M &= \frac{1}{T^2} \sum_{t=1}^{T} \sum_{r=1}^{T} \hat{u}_{t-r}\left[z_{t-r}x_{t-r} + z_{t-r}y_{t-r}\right] \\
\text{and} \quad X &= \left(y_{t-1}, x_t, x_{t-1}\right)
\end{align*}
\]

The estimators \( \left(\alpha, \hat{\beta}_1, \hat{\beta}_2\right) \) can be obtained using most software packages which include a Generalised Instrumental Variable option. The weighting matrix can be generated using a Two-Stage Least Squares Regression.
Having estimated the models in this way, the simulation procedure then uses the method of bootstrapping to establish whether the resultant estimators are consistent. Under this method, the residuals from the initial IV estimation are resampled, and the resampled residuals and the initial parameter estimates are combined to reconstruct the data $y_t^*$. These $y_t^*$ are then used to re-estimate the parameters of interest, $\alpha^*$ and $\beta^*$. Repeating this procedure gives the bootstrapped distribution of $\alpha^*$ and $\beta^*$.

**Results**

**Theoretical Results**

When the serial correlation takes a moving-average form, these instruments are all strictly valid, and the resulting IV estimators will be consistent. When the errors take an autoregressive or a fractionally integrated form, these instruments are invalid, so, strictly speaking, the IV estimators should be inconsistent. The presence and extent of the estimators’ inconsistency was derived and shown to be a complex function of the following four factors:

1. The structure of the serial correlation (AR, MA or FI)
2. The values of the parameters of interest: $\alpha$, $\beta_1$ and $\beta_2$
3. The value of the nuisance parameter, $\rho$
4. The process generating the exogenous variable $x_t$
5. The ratio of the error variance to the explanatory variable variance, or the signal-to-noise ratio (related to $R^2$ measures).

Having dissected the cause of estimator inconsistency into a function of these factors, it is then possible to vary each factor sequentially, to determine the impact that each one has upon the IV estimators. Since the IV estimators can rely on the strength of the instruments and the model goodness-of-fit, the asymptotic independence of the errors and the instruments is nearly always well-approximated for shorter lag lengths than that required to approximate the asymptotic independence of the instruments and the variables.

For completeness, the models were simulated for the entire range of permissible parameter values and combinations, to ensure that the estimation method is robust to variations in the true model, because in practice none of the above factors are known. The following ranges of parameter values were considered:
Table 1: Range of Parameter Values Considered

<table>
<thead>
<tr>
<th>Error structures</th>
<th>AR, MA, FI</th>
</tr>
</thead>
<tbody>
<tr>
<td>α values</td>
<td>0.1, 0.2... 0.9</td>
</tr>
<tr>
<td>β values</td>
<td>0.5, 0.8, 1.3</td>
</tr>
<tr>
<td>ρ values</td>
<td>0.1, 0.2... 0.9</td>
</tr>
<tr>
<td>$R^2$ values</td>
<td>0.5, 0.8, 0.9</td>
</tr>
</tbody>
</table>

These values are considered to be representative of those most commonly encountered in practical applications. Only positive values were considered because the results can be shown to be symmetrical, so these results also hold for negative parameter values.

Simulation Results

Having simulated the models under the various parameter constellations given above, the next step is to estimate them and determine whether the resultant estimators are consistent.

Consistency measurements are defined with respect to this bootstrapped distribution of the estimated parameters, as follows:

- **Criterion 1 (STRONGER)** The true parameter value equals, or is within an 0.05 percentile interval of, the mean of the bootstrap distribution (indicated below by $A^*$).

- **Criterion 2 (WEAKER)** The true parameter value lies in an 0.95 percentile interval.

Selected experiment results are shown in the tables below. Each case where the first criterion is not met is indicated with a star, and those where the second are not met are indicated by crosses.
Robust Estimation of Dynamic Models with Unmodelled Errors

### Table 2: Estimated $\alpha$, for MA Errors where true $\alpha = 0.5$

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$R^2 = 0.5$</th>
<th>95% interval</th>
<th>$R^2 = 0.5$</th>
<th>95% interval</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean (should = 0.5)</td>
<td></td>
<td>Mean (should = 0.5)</td>
<td></td>
</tr>
<tr>
<td>$\beta = 0.2$</td>
<td>$p = 0.2$</td>
<td>0.48</td>
<td>(0.36, 0.6)</td>
<td>0.53</td>
</tr>
<tr>
<td></td>
<td>$p = 0.6$</td>
<td>0.50</td>
<td>(0.33, 0.61)</td>
<td>0.51</td>
</tr>
<tr>
<td></td>
<td>$p = 0.9$</td>
<td>0.53</td>
<td>(0.31, 0.75)</td>
<td>0.49</td>
</tr>
<tr>
<td>$\beta = 0.2$</td>
<td>$p = 0.2$</td>
<td>0.48</td>
<td>(0.39, 0.57)</td>
<td>0.52</td>
</tr>
<tr>
<td></td>
<td>$p = 0.6$</td>
<td>0.50</td>
<td>(0.41, 0.60)</td>
<td>0.50</td>
</tr>
<tr>
<td></td>
<td>$p = 0.9$</td>
<td>0.53</td>
<td>(0.37, 0.70)</td>
<td>0.49</td>
</tr>
</tbody>
</table>

### Table 3: Estimated $\alpha$, for AR Errors where true $\alpha = 0.5$

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$R^2 = 0.5$</th>
<th>95% interval</th>
<th>$R^2 = 0.5$</th>
<th>95% interval</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean (should = 0.5)</td>
<td></td>
<td>Mean (should = 0.5)</td>
<td></td>
</tr>
<tr>
<td>$\beta = 0.2$</td>
<td>$p = 0.2$</td>
<td>0.50</td>
<td>(0.31, 0.70)</td>
<td>0.52</td>
</tr>
<tr>
<td></td>
<td>$p = 0.6$</td>
<td>0.50</td>
<td>(0.36, 0.65)</td>
<td>0.52</td>
</tr>
<tr>
<td></td>
<td>$p = 0.9$</td>
<td>0.92 *</td>
<td>(0.79, 1.06) †</td>
<td>0.62 *</td>
</tr>
<tr>
<td>$\beta = 0.2$</td>
<td>$p = 0.2$</td>
<td>0.49</td>
<td>(0.34, 0.66)</td>
<td>0.52</td>
</tr>
<tr>
<td></td>
<td>$p = 0.6$</td>
<td>0.52</td>
<td>(0.40, 0.67)</td>
<td>0.52</td>
</tr>
<tr>
<td></td>
<td>$p = 0.9$</td>
<td>0.68 *</td>
<td>(0.59, 0.78) †</td>
<td>0.51</td>
</tr>
</tbody>
</table>

Several facts must be noted from these tables. Firstly, the IV method is consistent for all cases when the errors have an MA structure. Secondly, the IV method is also consistent for the majority of cases where the errors have an AR structure, except when the persistence of the serial correlation in the errors is extremely high ($\rho = 0.9$) and the $R^2$ value is particularly low.

The implication of these results is that the method of IV estimation is robust to a surprisingly large range of parameter values - which advocates its potential use in a wide range of modelling circumstances. Further evidence is provided graphically in the following two panels, which explicitly indicate the effect of varying the persistence of the serial correlation. These graphs represent the bootstrapped distribution of $\alpha$ across the x and z axes, for different values of $\rho$, which are represented across the y axis. Consistency is implied when the bootstrapped distribution is centred over the true value $\alpha = 0.3$. The model here is parameterised so as to have relatively weak instruments, and poor model fit, and we are interested in the effect of varying the serial correlation persistence for such a model (i.e. the effect of moving along the y axis).
For the moving-average case presented in the left panel, the distribution is centred on the true value $\alpha = 0.3$ for all $\rho$, so consistent estimators are obtained regardless of the value of $\rho$. For the autoregressive case on the right, the distribution is centred on the true value only when $\rho < 0.6$. When $0.6 < \rho < 0.8$, the distribution is no longer centred on the true value, which classes the estimators as inconsistent by the first criterion, although it is still within a 0.95 percentile interval around the true value, so the second criterion is met. Only for very strong serial correlation, where $\rho > 0.8$, does the estimate entirely depart from the true value. This is one of the areas of the parameter space for which the instrumental variable technique is unable to perform effectively.

We now look at the effect of changing the model fit, for the case with autoregressive errors:
This time, the bootstrapped distribution is centred on the true value for all values of $\alpha$, indicating that the estimators are consistent no matter how strong the serial correlation in the errors. This important result highlights the key advantage of the instrumental variable approach: its ability to effectively "swamp" the serial correlation problems in the errors by using very strong instrument conditions.

These truncated results indicate the methods used to assess the IV estimators. These were applied across the entire parameter spectrum, and the general results are presented in the following list. Consistent estimation by IV is possible if any one of the following conditions hold:

1. The errors have a moving average structure
2. The serial correlation is less persistent ($\rho < 0.6$)
3. The parameters of interest are high, such that the instruments are stronger, and $\rho < 0.9$
4. $R^2 > 0.5$.

This leaves very few cases where the IV method fails to provide consistent estimates. Exceptions arise when the instruments are very weak, the model fit is poor and the persistence of the serial correlation in the errors is very high. These exceptional cases were found to comprise less than 7% of all cases examined, and these cases (representing very high serial correlation, very poor model fit, etc.) are uncommon in practical applications. This failure rate must be compared to existing methods of autocorrelation correction such as the Cochrane-Orcutt procedure, which has been shown in the literature to fail in all but a very select, specific and practically obsolete category of static models.

**Implications for APRA**

A key element of APRA’s supervisory role is the statistical analysis of institutions’ failure probabilities, using the Probability and Impact Rating System. This translates the calculated risk from a linear risk score into a non-linear failure probability index, using regression methods based on the long term average relationship between risk ratings and incidences of default. In fact, the construction of many prudential standards, benchmarks and minimum requirements are based on long run average relationships between various financial indicators and failure rates. The pitfalls of analysing long term time series data have been indicated within this report, and the likelihood that such data suffers from serially correlated errors is high. The adoption of the instrumental variable technique may benefit the accuracy of evaluating risk, the determination of supervisory stance and the allocation of resources.

More generally, time series analysis is integral to bodies such as the Reserve Bank of Australia, as well as to APRA, in evaluating economic trends and forecasting. APRA relies on contextual economic information when assessing risk.
The advantage of the instrumental variable method is that it is easily implemented through most software packages. The results contained in this paper are not proposing a new method, but rather aimed at proving the validity of an existing method in a new time series context. This is hoped to provide practitioners with confidence to apply a familiar method, and also to eradicate some of the most common dangers of dynamic estimation.

**Conclusion**

The approach currently employed to deal with serial correlation in the errors of dynamic models is to proceed using instrumental variable estimation if the errors take a moving average form, and use an autocorrelation correction method, such as the Cochrane-Orcutt procedure. Conditioning on the type of serial correlation is unsatisfactory, since in practice this is not known and serial correlation may arise from a number of different causes. Therefore, there is a need for an estimation method that is robust to any type of serial correlation. The arguments and simulations contained in this research paper are intended to show an instrumental variable estimation approach provides consistent estimation under surprisingly mild conditions. The parameter constellations for which the instrumental variable approach fails are much smaller than any existing method: a result which lends support to the adoption of IV estimation as a viable and preferable alternative to current estimation practices.