Working Paper

On the Coherence of VAR Risk Measures for Levy Stable Distributions

Wilson Sy — issued May 2006
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Summary

The Value-at-Risk (VaR) risk measure has been widely used in finance and insurance for capital and risk management. However, in recent years it has fallen somewhat out of favour due to a seminal paper by Artzner et al. (1999) who showed that VaR does not in general have all the four coherence properties which are desirable for a risk measure. In particular, the violation of the sub-additive property discourages diversification and is counter-intuitive to risk finance. In this paper, it is proved (Theorem 3.1) that VaR for independent Levy-stable random variates is a coherent risk measure being translational invariant, monotonic, positively homogeneous and sub-additive. That is, Levy-stable distributions are VaR coherent. As Levy-stable distributions are a rich class of probability distributions, the VaR risk measure may still have widespread applications. A brief comparative discussion is also given for L-stable variates for the expected shortfall risk measure.
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Introduction

The Value-at-Risk (VaR) risk measure has been widely used in finance and insurance for capital and risk management. For example, in the new Basel accord (BIS, 2005) banks will have the opportunity to use the internal modelling methods with the VaR approach, as a standard risk measure, to assess capital requirements. The incentive for using internal modelling methods rather than the alternative simpler standardised method is the possibility of getting a reduction in capital requirement due to the potential ability of these methods to quantify the benefits of risk diversification.

The social and financial benefits from risk diversification are even more fundamental to insurance, where risk pooling is the raison d’etre for the industry. In catastrophe insurance (Woo, 2002), capital reserves are calculated from the Probable Maximum Loss (PML) for each catastrophic event using the VaR approach.

Artzner et al. (1999) show that in general VaR does not have all the desirable coherent properties for a risk measure and in particular it may not have the sub-additive property (see below) in some applications. Having the sub-additive property is essential to showing benefit from diversification which is fundamental to capital and risk management in finance.

Coherence for a risk measure can be defined by the following four axioms. Given a risk measure \( \rho \) on a set of risks \( \mathcal{R} \), the risk measure is coherent if it satisfies the following axioms on the properties of \( \rho \):

(A1) Translational invariant: for all \( X \in \mathcal{R} \) and all real numbers \( c \), \( \rho(X+c) = \rho(X) + c \).

(A2) Monotonic: for all \( X_1, X_2 \in \mathcal{R} \), \( X_1 > X_2 \Rightarrow \rho(X_1) > \rho(X_2) \).

(A3) Positively homogeneous: for all \( X \in \mathcal{R} \) and all \( \lambda \geq 0 \), \( \rho(\lambda X) = \lambda \rho(X) \).

(A4) Sub-additive: for all \( X_1, X_2 \in \mathcal{R} \),

\[ \rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2) \]

Since the seminal paper of Artzner et al. (1999), many others have shown more examples of inconsistencies of the VaR approach in finance and insurance (e.g. Evans, 2001) in breakdown of the sub-additive property or in harm arising from diversification.

The disenchantment with VaR had reached such a level that a whole issue of the Journal of Banking and Finance was devoted to criticising VaR (Szego, 2002) and to finding alternative approaches (e.g. Acerbi and Tasche, 2002). The editorial of that issue of the journal concludes by exhorting its readers to ‘do your best to convince the regulators to go back to the drawing board’.

On the other hand, it is well-known that the VaR risk measure is coherent for the Normal (Gaussian) random variates and in particular it is sub-additive for this class of distribution. The sub-additive property is so important in finance that normal distributions are often used in practice to fit non-normal data, particularly in portfolio applications such as CreditMetrics (Gupton et al., 1997), despite the danger of possibly underestimating risk arising from fat-tailed distributions of the actual data.

In this paper, the range of validity of VaR risk measure is improved by extending the coherence of VaR risk measure to a larger class of probability distributions. It is proved that the VaR risk measure is coherent beyond normal variates to include a class of independent Levy-stable (L-stable) variates. As the L-stable distributions are a rich class of probability distributions, including those with fat-tails, the VaR risk measure may have wider applications than previously thought.

---

1 Internal modeling methods include the Internal Rating Based (IRB) method for credit risk and the advanced measurement method (AMA) for operational risk for Basel II.

2 To encourage financial institutions to have a clearer understanding of their own risks, regulatory policy is set so that the prescriptive or standardised methods usually have more conservative (or higher) capital requirements than would be obtained from a more detailed understanding of the actual risks.
Since 1950s Mandelbrot (1997) has demonstrated the importance of Levy-stable distributions for explaining empirical data in finance and economics. In particular, he has emphasised the role played by Levy-stability in the scaling of random market prices over time. It is shown here that Levy-stability of the random variates is important in deriving the sub-additive property (A4) for coherence.

The sub-additive property (A4) is seen as a condition on a random multivariate which is the sum of univariates. Most recent research in risk management has tended to focus on the marginal dependences of multivariates, especially in their tails (e.g. Embrechts and Puccetti, 2006). In contrast, the focus in this paper is on sums of independent L-stable univariates and their statistical properties.

Let the potential liability or loss from a portfolio of risks be represented by a random variate \( X \), which is a linear combination of \( n \) independent random L-stable variates

\[
X = \sum_{i=1}^{n} X_i,
\]

where \( X_i \) is a random L-stable variate for the \( i \)th risk. The sum of variates can be extended to a weighted sum of variates, but the simpler combination (2.1) is used here for clarity, without loss of generality.

The random variates representing risks are assumed to be independently, but not identically distributed. The L-stable distributions have the property that a linear combination of independent random variates from a family of \( \alpha \)-stable distributions is a random variate with a distribution in the same \( \alpha \)-stable family. L-stability is an important property for the portfolio approach, because there are known and useful statistical properties of multivariates which are sums of stable variates (e.g. Feller, 1971; Zolotarev, 1986; Nolan, 2006). The Levy \( \alpha \)-stable probability density functions in the random variate \( x \) are given by

\[
f(x; \alpha, \beta, \gamma, \mu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(t) \exp(-itx) dt
\]

where the Greek letters are parameters and the characteristic function is defined (for algebraic convenience) by

\[
\phi(t) = \exp \left[ \mu t - \gamma|t|^\alpha (1 - i\beta \Phi / |t|) \right]
\]

The function \( \Phi \) is given by

\[
\Phi = \tan(\pi \alpha / 2), \quad \alpha \neq 1,
\]

\[
= -(2 / \pi \log |t|) \quad \alpha = 1.
\]

The random variate of this distribution is stable for \( 0 < \alpha \leq 2, -1 \leq \beta \leq 1, \gamma \geq 0 \) and \( \mu \) given on the real line.

There are no known closed form solutions for the probability density function (pdf) defined by the integral in (2.2) for general values of the parameters. This has been a barrier to wider use of L-stable distributions. But with advances in computers and numerical methods, many of the tasks which need to be done in applications such as calculating probabilities can be done without formulas (Mandelbrot, 1997; Garcia et al., 2004). For special values, pdf (2.2) has close forms and reduces to the Normal, Cauchy and Levy distributions.

Note that \( \gamma \) is a spread or width measure for the distribution. The parameter \( \beta \) determines skewness, whereas the parameter \( \mu \) determines the location of the distribution.

Assume that the L-stable variates in equation (1) representing catastrophic insurance risks are independent and each has different parameters \( \beta_i, \gamma_i \) and \( \mu_i \) \( (i = 1, 2, 3 \ldots n) \), but the same \( \alpha \).

It follows immediately from the property of L-stability and equation (2.2) that the portfolio of risks \( X \) given by equation (2.1) is an L-stable variate with the same stable \( \alpha \) and with other parameters given by

\[
\beta = \sum_{i=1}^{n} \beta_i \gamma_i^{\alpha} / \gamma^\alpha,
\]

\[
\gamma^\alpha = \sum_{i=1}^{n} \gamma_i^{\alpha},
\]

\[
\mu = \sum_{i=1}^{n} \mu_i,
\]

for the chosen parameterization (Nolan, 2006) of a stable law used in equation (2.3). Other parameterizations give similar, but more complex relationships.
Coherence of VaR risk measure

The VaR risk measure $\rho$ for a random capital loss variate $X$ at fixed probability $p$ (say $p = 0.995$ for some Basel applications) is defined here as

$$
(3.1) \quad \rho(X) = \rho(X, p) = \inf \{ \lambda \mid P \{ X \leq \lambda \} \geq p \}.
$$

The probability measure of equation (3.1) $P[\cdots]$ is defined as follows. Let the risks in equation (2.1) be represented by elements $X$ of $\Lambda$ in a probability space $(\Omega, \mathcal{A}, P)$ where $\Omega$ is the real line $t$, then the probability measure $P[\cdots]$ is given by the cumulative probability function of equation (2.2):

$$
(3.2) \quad P [ X \leq x ] = \mathbb{P}(X, \alpha, \beta, \gamma, \mu) = \int_{-\infty}^{x} f(u, \alpha, \beta, \gamma, \mu) du.
$$

The purpose of this paper is to prove the following theorem.

**Theorem 3.1**

The Value-at-Risk risk measure (3.1) for independent L-stable random variates with probability density functions (2.2) $(1 \leq \alpha \leq 2)$ is a coherent risk measure being translational invariant, monotonic, positively homogeneous and sub-additive, as defined by axioms (A1)-(A4).

**Proof**

(A1) Translational invariant:

Translation of the VaR risk measure is tantamount to shifting the location of the probability density function. From equations (2.2) and (2.3), it follows that

$$
(3.3) \quad f(x, \alpha, \beta, \gamma, \mu) = f(x-\mu, \alpha, \beta, \gamma, 0).
$$

This implies that the capital risk measure defined in equation (3.1) is translational invariant i.e. $\rho(X+c) = \rho(X) + c$.

(A2) Monotonic:

Excluding the case $\alpha = 1$ and $\beta \neq 0$, a change of the $t$ variable in equations (2.2) and (2.3) can be used to show

$$
(3.4) \quad f(x, \alpha, \beta, \gamma, \mu) = f(x/\gamma, \alpha, \beta, 1, \mu/\gamma).
$$

This relationship can be used to prove from (3.2)

$$
(3.5) \quad F(x, \alpha, \beta, \gamma, \mu) = F((x-\mu)/\gamma, \alpha, \beta, 1, 0).
$$

On dropping common parameters $\alpha$ and $\beta$, it can be written as

$$
(3.6) \quad F(x, \gamma, \mu) = F((x-\mu)/\gamma, 1, 0).
$$

Defining for convenience $\bar{\rho}$ as the risk measure for a centred L-stable random variate with unit spread $(\gamma = 1)$ i.e. $F(\bar{\rho}, 1, 0) = p$ or $\rho = F^{-1}(p)$. Then equations (3.1), (3.2) and (3.6) then imply

$$
(3.7) \quad \rho(X) = \mu + \gamma \bar{\rho}.
$$

The L-stable random variates are ordered by the spread parameter $\gamma$ so that $X_i > X_j$ if and only if $\gamma_i > \gamma_j$, with other parameters constant. It follows that $\gamma_i < \gamma_j$ implies $\rho(X_i) > \rho(X_j)$ i.e. this proves the risk measure is monotonic: $X_i > X_j \Rightarrow \rho(X_i) > \rho(X_j)$.

(A3) Positively homogeneous:

Since a random variate $\lambda X$ for constant $\lambda > 0$ scales the variate both in the location and the spread parameters: $\lambda \mu$ and $\lambda \beta$, it follows from (3.7) that the risk measure is positively homogeneous:

$$
(3.8) \quad \rho(\lambda X) = \lambda \rho(X).
$$

(A4) Sub-additive:

Finally, it follows from equations (2.5) and (2.6) that the sum of two independent L-stable variates preserves skewness as $\beta_1 = \beta_2 = \beta$ and applying equation (3.7) and the L-stability property in equations (2.6) and (2.7) for a mixture of two L-stable variates, one finds

$$
(3.9) \quad \rho(X_i + X_j) = \mu + \gamma \bar{\rho} = \mu_i + \mu_j + (\gamma_i^{1/\alpha} + \gamma_j^{1/\alpha})^{1/\alpha} \bar{\rho}.
$$

It follows from inequality for $\alpha \geq 1$ that

$$
(3.10) \quad (\gamma_i^{1/\alpha} + \gamma_j^{1/\alpha})^{1/\alpha} \leq \gamma_i + \gamma_j, \quad \text{which can be used to directly show with equations (3.7) and (3.8), that the risk measure is sub-additive: } \rho(X_i + X_j) \leq \rho(X_i) + \rho(X_j).
$$

This concludes the proof of the theorem. Note the theorem has been proved for the simplest algebraic parameterisation of the L-stable distribution equation (2.3) corresponding to $k = 1$ in Nolan (2006) or (A) parameterization in Zolotarev (1986). The proof for Nolan’s $k = 0$ or Zolotarev’s (M) parameterization proceeds in an analogous, but slightly more complicated, fashion with the result that the equation corresponding to (2.7) has more terms and that $\bar{\rho}$ in equation (3.7) needs to be adjusted by constants $\beta \tan(\pi \alpha / 2)$ and $2\beta \log \gamma / \pi$ for the case $\alpha \neq 1$ and $\alpha = 1$ respectively.
It should be noted that proof the sub-additive property (A4) depends critically on L-stability properties (2.6) and (2.7) to obtain the step in equation (3.8) and that the sub-additive property is not obtained for $0 < \alpha < 1$, since the inequality $(\gamma_1^{\alpha} + \gamma_2^{\alpha})^{1/\alpha} \leq \gamma_1 + \gamma_2$ is not satisfied (e.g. the coin-tossing case with $\alpha = 0.5$, $\beta = 1$, $\gamma = 1$, $\mu = 0$).

In the asymptotic limit (i.e. only for large values of the variable), the sub-additive property has been shown by Garcia et al. (2005) for some ($\beta < 1$) stable distributions and by Danielsson et al. (2005) for fat-tailed distributions. Unlike these studies in the above proof, the sub-additive property has been shown to hold exactly and generally for ($\alpha \geq 1$) L-stable distributions.

Observe that setting $\alpha = 2$ recovers the Normal (Gaussian) case, which has a variance $\sigma^2 = 2 \gamma^2$ expressed in L-stable notation. It follows from equation (2.6) the well-known result for a mixture of independent Normal variates defined by equation (2.1) is recovered

\[
(3.9) \quad \sigma^2 = \sum_{i=1}^{n} \sigma_i^2.
\]

From above discussion, it is clear that the VaR risk measure is proportional numerically to the spread $\gamma$ and hence it is proportional to the standard deviation or the volatility (in finance) $\sigma$ for the Normal case. However, $\sigma$ cannot be used as a coherent risk measure because it does not have translational invariance (A1). There does not appear either to be any simple algebraic function of $\sigma$ and $\mu$ which could be used as a coherent risk measure, as at least one of the coherence axioms (A1)-(A4) is violated. Los (2004) noted that equations (2.6) and (3.9) would indicate the existence of a diversification effect.
Diversification benefit

It follows from the sub-additive property that there is social benefit from diversification through risk pooling. Since our risk measure is in capital value, the social surplus can be quantified in capital terms for a portfolio of L-stable risk variates. If $\rho(X_i)$ is abbreviated to $\rho_i$, then a simple generalisation of the discussion in equation (3.8) shows that the social surplus can be written as

$$S = \sum_{i=1}^{n} \rho_i - \rho.$$

The first term on the right hand side of equation (4.1) is the total capital requirement for when the risks are held individually and separately, while the second term is the capital requirement for the portfolio as a whole, with the benefit of diversification. The social surplus $S$ is the reduction in capital requirement from risk pooling and it is positive due to a mathematical inequality for $\alpha > 1$ in equation (3.8), provided the L-stable risk variates have the same $\alpha$ and $\beta$. There is no diversification benefit for the case where $\alpha = 1$, corresponding to the Cauchy distribution with extremely fat tails.

There is also no diversification benefit for completely correlated Normal variates as is well-known in standard multivariate probability. As independence of the L-stable variates has been assumed in this paper, no corresponding definitive comment can be made about a generalisation of the sub-additive property (A4) to include dependences without a fuller multivariate probability theory of L-stable distributions, which is an active research topic beyond the scope of this paper.
Discussion

This paper was initially motivated by a desire to extend the validity of the VaR risk measure to a wider class of probability distributions, which could accommodate empirical data in practical applications. There appears to be no single risk measure, VaR or its alternatives which is coherent with respect to all arbitrary probability distributions. (Expected Shortfall risk measures are briefly discussed in the Appendix.) On the contrary, this paper suggests a particular risk measure is coherent only with respect to particular class (or classes) of probability distributions.

The result of this paper can be stated in another way: Levy-stable distributions with $\alpha \geq 1$ have the property of VaR coherence and expected shortfall coherence (see Appendix). In other words, coherence with respect to a particular risk measure is viewed as a property of certain probability distributions, i.e. a coherent risk measure $\rho$ needs to refer to a probability space $(\Omega, \mathcal{A}, P)$ as defined in equation (3). Therefore, in general, a risk measure is coherent only with respect to given probability spaces.

The coherence of VaR risk measure requires at least non-divergent first moment which is the case for Levy-stable distributions only with $\alpha \geq 1$. The sub-additive property of the VaR risk measure is a consequence of the L-stability of the probability distributions through equation (2.5)-(2.7) and the convergence of their first moments. In the numerous examples (e.g. Szego, 2002 and Evans, 2001) where the VaR risk measure violates the sub-additive property the probability distributions do not have the necessary properties.

Levy-stable distributions are natural extensions of the Normal (Gaussian) distribution. The Log-Normal distribution on the positive real line is obtained from the Normal distribution on the real line by a simple logarithmic transformation of the random variate. Similarly, a Log-stable distribution can be defined from a stable distribution by a variable transformation. This class of transformed distributions further enhances the possible usefulness of the Levy-stable distributions.

In conclusion, unless the VaR risk measure is valid for a sufficiently wide class of probability distributions, its application to many areas of banking and insurance could be questioned and in particular the rationale for the new Basel accord could be undermined. This paper has proved that the Levy-stable distributions may hopefully constitute such a wide class of probability distributions for practical applications.
Appendix — Expected shortfall

In this appendix, a brief discussion is given on expected shortfall (ESf) as an alternative risk measure to Value-at-Risk (VaR), for Levy-stable distributions. As Acerbi and Tasche (2002) show, there are several definitions of expected shortfall and many do not generally lead to coherent risk measures. Two definitions which are coherent for Levy-stable variates are considered here.

Consider firstly the definition of expected shortfall for continuous distributions given by

(A.1) \[ \sigma(X,p) = \frac{1}{1-p} \int P(X,u) du \]

In this expression the VaR risk measure \( \rho(X,u) \) is defined as in equation (3.1). With this definition for expected shortfall, the coherence properties of being translational invariant, positively homogeneous and monotonic for \( \sigma(X,p) \) follow immediately from the coherence of \( \rho(X,u) \).

For brevity, define functions

(A.2) \[ F(p) = F(X; p) = \rho(X,p) + \rho(Y,p) - \rho(X,Y,p) \]

(A.3) \[ G(p) = G(X; Y; p) = \sigma(X,p) + \sigma(Y,p) - \sigma(X,Y,p) \]

It is assumed that the expected shortfall risk measure is defined for the sum two random variates \( X+Y \) (see below). If \( F(p) \geq 0 \) for \( p \in [0,1] \) then since

(A.4) \[ \frac{1}{1-p} \int F(u) du \geq 0 \quad \text{for} \quad p \in [0,1] \]

\( F(p) \geq 0 \) for \( p \in [0,1] \) \( \Rightarrow G(p) \geq 0 \) for \( p \in [0,1] \); that is, sub-additive VaR \( \rho(X,u) \) implies sub-additive expected shortfall \( \sigma(X,p) \) given by (A.1). The following theorem has been proved.

Theorem A.1

The expected shortfall risk measure defined by equation (A.1) for independent L-stable random variates with probability density functions (2.2) \((1 \leq \alpha \leq 2)\) is a coherent risk measure.

Consider next another definition of the expected shortfall given by

(A.5) \[ \sigma(X,p) = \int x f(x) dx/ \int f(x) dx = \frac{1}{1-p} \int x f(x) dx \]

In this expression, the random variate \( X \) has probability density function of \( f(x) \) for a loss or liability function, which is required to have convergent first moment.

The generalisation of (A.5) to the sum of two random \( X+Y \) would require the evaluation of a multiple integral

(A.6) \[ \sigma(X+Y,p) = \int z f(z-y) f(y) dy dz \]

In general, useful statistical properties such as those possessed by Normal variates (even with dependencies) and independent Levy-stable variates are needed to proceed further formally.

Provided \( X \) and \( Y \) are independent and identically distributed (iid) stable random variates, it can be shown from convolution properties of the distributions that the sum of \( X \) and \( Y \) has expected shortfall

(A.7) \[ \sigma(X+Y,p) = \frac{1}{1-p} \left\{ \int x f(x) dx + \int y f(y) dy \right\} \]

From the monotonic property of the VaR risk measure for stable variates, \( \rho(X+Y,p) \geq \rho(X,p) \) and \( \rho(X+Y,p) \geq \rho(Y,p) \), which implies from equation (A.6) that

(A.8) \[ \sigma(X+Y,p) \leq \frac{1}{1-p} \int x f(x) dx + \frac{1}{1-p} \int y f(y) dy \]
An often cited criticism of the VaR risk measure in favour of expected shortfall is that VaR does not take into account the extent of the loss, once the VaR threshold is breached, whereas expected shortfall does. It is therefore interesting to evaluate the consequences of the differences as there is a prima facie option of adopting expected shortfall, since VaR coherence has been proved to imply expected shortfall coherence.

Consider a random loss variate \( X \) with a unit Normal probability density function \( f(x) \). Define a VaR risk measure at probability \( p \) by

\[
\rho(X,p) = x_p \quad \text{where} \quad 1 - p = \int_{-\infty}^{x_p} \exp\left(-\frac{x^2}{2}\right) dx
\]

It is easy to show from (A.5) and (A.8) that

\[
\sigma(X,p) = \frac{1}{\sqrt{2\pi}} \int_{x_p}^{\infty} \exp\left(-\frac{x^2}{2}\right) dx
\]

Note that in equation (A.8), constant skewness parameter \( \beta \) has been assumed. Otherwise, an inequality such as \( \rho(X+Y,p) \geq \rho(X,p) \) has not been proved yet to be valid, since the risk measure for unit spread \( \rho \) in equation (3.7) will change for different values of \( \beta \). The impact of heterogenous \( \beta \) on the risk measure needs further investigation.

In their proof of the coherence of the expected shortfall risk measure for general distributions, Acerbi and Tasche (2002) do not require VaR coherence, which, in contrast, has been essential here in the above proofs of expected shortfall coherence for L-stable variates. Some detailed knowledge of the properties and restrictions on the probability density function of the sum of two random variates have been necessary in equation (A.6) to evaluate and compare expected shortfalls.

The expression (A.5) with the probability factor \( 1/(1-p) \) replaced by unity (or some other reference constant factor) is the definition for the conditional Value-at-Risk (CVaR), which is selected, in preference to VaR, by Stoyanov et al. (2004) to measure portfolio risk in the L-stable case, because they assume CVaR is a coherent risk measure and by implication VaR is not. Note that CVaR is smaller than expected shortfall by the same factor \( 1/(1-p) \), which can be very large for extreme values \( p \to 1 \). In such situations, CVaR can be much smaller than VaR (see below).
Table 1 Sample values of VaR and ESf for the unit normal distribution

<table>
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<th>VaR</th>
<th>Probability</th>
<th>ESf</th>
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<td>1.0</td>
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<td>4.0</td>
<td>0.9999683</td>
<td>4.2236</td>
<td>1.0559</td>
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The situation is different for fat-tailed distributions. In the asymptotic limit of Levy-stable distributions (Gorey and Sy, 2006), the relationship corresponding to (A.10) is

\[ \frac{\sigma(X,p)}{\rho(X,p)} = \frac{\alpha}{\alpha - 1} \]  

(A.11)

The tail exponent is assumed to be in the range \(1 \leq \alpha \leq 2\). Hence expected shortfall is larger than VaR risk measures for non-Normal variates by a factor of between two and infinity. Moreover, to lowest order of approximation, there is no tendency for the two risk measures to converge as probability approaches unity (as in the Normal case).

This implies that capital requirements based on expected shortfall calculations would be significantly higher (more than double) than that from VaR calculations for non-Normal distributions, found often in practical applications such as catastrophe insurance. This has important implications for the regulation of the insurance industry, where VaR calculations may underestimate potential catastrophic losses. Capital requirement based on VaR with probability \(p\) would indicate failure with probability \(1 - p\), whereas capital requirement based on expected shortfall would indicate survival with at least probability \(p\).


